VECTOR BUNDLES AND GAUGE THEORY
UW MATH 865 – SPRING 2022

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Part I. Topological classification of vector bundles

1. Definition, first examples, bundle morphisms (1/25)

1.1. **The definition.** Let $\pi : E \twoheadrightarrow X$ be a surjective map between topological spaces. (A map will always be assumed to be continuous.)[1]

Given an open set $U \subset X$, a **section** of π over U is a map $s : U \to E$ such that

$$\pi \circ s = \mathbf{1}_U.$$

Equivalently, s must satisfy

$$s(x) \in E_x = \pi^{-1}(x)$$

for each $x \in U$. In case $U = X$, s is called a **global section**.

We write[2]

$$\Gamma(U, E) = \{\text{sections of } \pi \text{ over } U\}.$$

These sets obviously fit together to form a sheaf over X, although we do not plan to use this fact in the short term.

Fix a base field $K = \mathbb{R}$ or $\mathbb{C}$. Many of the results will also apply for $K = \mathbb{H}$, but some care may be required.

Definition 1.1. We say that $\pi : E \to X$ is a **vector bundle** over K of rank $r \in \mathbb{N}$ if:

(1) Each fiber E_x is endowed with operations + and $\cdot$ giving the structure of a vector space over K of dimension r, and these operations are continuous in the subspace topology on $E_x \subset E$.

(2) For each $x_0 \in X$, there exists an open neighborhood $U \ni x_0$ and a **local frame** of sections $\{e_\alpha\}_{\alpha=1}^r$ over U, such that the map

$$U \times K^r \to \pi^{-1}(U) \subset E$$
$$(x, a^1, \ldots, a^r) \mapsto a^1 e_1(x) + \cdots + a^n e_n(x)$$

is a homeomorphism.

1.2. **Terminology.** E = "total space," X = "base space," $(U, \{e_\alpha\})$ = "local trivialization."

Vector bundle of rank 1 = "line bundle" (Note: real and complex line bundles have very different properties.)

Given $s : V \to E$ and $U \subset V$, the **restriction** of s is given by the composition

$$s|_U : U \hookrightarrow V \xrightarrow{s} E.$$

[1]Thanks to Connor Simpson for providing the latex draft of these notes after each class!

[2]As of now, Γ refers only to the space of continuous sections. For smooth or holomorphic bundles, it will denote the space of smooth or holomorphic sections.

If $\{e_\alpha\}$ is a local frame for E over U, the **local components** of of s,

$$s^\alpha : U \cap V \to K, \quad \alpha = 1, \dots, r,$$

are defined by the rule

$$s|_{U\cap V} = \sum_\alpha s^\alpha e_\alpha.$$

These are defined by the composition

$$s^\alpha : U \cap V \xrightarrow{s|_{U\cap V}} \pi^{-1}(U \cap V) \xrightarrow{\sim} (U \cap V) \times K^r \xrightarrow{\alpha} K,$$

where the second arrow is the inverse of the homeomorphism defined by the frame $\{e_\alpha\}$, according to item (2) of Definition 1.1. Hence, the local components are always continuous.

Lastly, given a complex vector bundle E of rank r, there is a canonical real vector bundle of rank $2r$ obtained by restricting the scalar field to $\mathbb{R} \subset \mathbb{C}$. This is referred to as the **underlying real bundle**. We shall sometimes abuse notation and refer to the underlying real bundle by the same letter E.

1.3. **Examples.**

- $\underline{K}^r = X \times K^r$, the **trivial bundle** of rank r.
- $E = [0,1] \times \mathbb{R}/\sim$, where $(0, v) \sim (1, -v)$ for all $v \in \mathbb{R}$. The base space is $X = S^1 = [0,1]/(0 \sim 1)$, with the obvious projection. This is the **Möbius bundle.** A local trivialization exists over any proper subset of S^1, but no global trivialization exists (exercise).
- The **tangent bundle** TS^2**,** defined by

$$\pi : TS^2 = \{(x, v) \in S^2 \times \mathbb{R}^3 \mid x \cdot v = 0\} \to S^2,$$

with the subspace topology from $S^2 \times \mathbb{R}^3$.

Given $x_0 \in S^2$, a local frame near x_0 can be defined as follows. Choose $x_1, x_2 \in \mathbb{R}^3$ such that $\{x_0, x_1, x_2\}$ form an orthonormal basis. Let

$$e_1(x) = x \times x_1, \qquad e_2(x) = x \times x_2.$$

Then these form a local frame on the open hemisphere $U = \{x \in S^2 \mid x \cdot x_0 > 0\}$.

Note that TS^2 can also be made into a $\mathbb{C}$-line bundle by the rule

$$i \cdot (x, v) = (x, x \times v)$$

This operation satisfies $i \cdot i = -1$, so gives a well-defined scalar multiplication by $\mathbb{C}$ (this is called an *almost-complex structure*).

1.4. **Bundle morphisms.** In what follows we shall often refer to a "vector bundle," with the fixed base field K understood.

Definition 1.2. Given two vector bundles E and F over X, a **bundle morphism** $\sigma : E \to F$ is a map such that the diagram

$$\begin{tikzcd} E \arrow[rr, "\sigma"] \arrow[dr] & \arrow[d, phantom, "\circlearrowright"] & F \arrow[dl] \\ & X & \end{tikzcd}$$

commutes, and $\sigma_x : E_x \to F_x$ is a linear map for each x. The morphism σ is an **isomorphism** if it has an inverse that is also a bundle morphism.

Proposition 1.3. *A bundle morphism $\sigma : E \to F$ is an isomorphism if and only if $\sigma_x : E_x \to F_x$ is an isomorphism for each $x \in X$.*

Proof. ($\Rightarrow$) This direction is trivial.

($\Leftarrow$) The existence of a set-theoretic inverse is obvious; we must show that it is a homeomorphism. The question is local, so we fix $x_0 \in X$ and local trivializations $(U, \{e_\alpha\})$ for E and $(V, \{f_\beta\})$ for F, with $x_0 \in U \cap V$.

For each α, let ${\sigma^\beta}_\alpha(x)$, $\beta = 1, \dots, r$, be the local components of $\sigma(e_\alpha)$ in the frame $\{f_\beta\}$ on $x \in U \cap V$; in other words, these are K-valued functions satisfying

$$\sigma(e_\alpha) = {\sigma^\beta}_\alpha f_\beta.$$

Since σ_x is assumed invertible for each x, we must have $|\det {\sigma^\beta}_\alpha(x)| > 0$. So $(\det {\sigma^\beta}_\alpha(x))^{-1}$ is continuous on $U \cap V$, and by Cramer's formula, ${\sigma^\beta}_\alpha$ has a continuous inverse. □

Remark 1.4. The same proof works in the category of smooth or holomorphic bundles.

Corollary 1.5. *A vector bundle E is* **trivial***, i.e. isomorphic to the trivial bundle $\underline{K}^r$, if and only if it admits a* **global frame,** *i.e., a set of global sections $\{e_\alpha\}_{\alpha=1}^r$ such that $\{e_\alpha(x)\}$ forms a basis for E_x for each $x \in X$.*

Proof. Simply define a bundle morphism

$$\underline{K}^r \to E$$
$$(x, a^1, \dots, a^r) \to a^1 e_1(x) + \dots + a^n e_n(x).$$

According to the previous proposition and the assumption, this is an isomorphism. □

1.5. **Transition functions and gluing.** There is another perspective on vector bundles. Given $E \to X$, pick a system of local trivializations $\{(U_a, \{e^a_\alpha\})\}$ such that $\cup_a U_a = X$. Define the **transition functions**

$$g_{ab} \in \operatorname{Map}(U_a \cap U_b, \mathrm{GL}(n, K))$$

by

$$e^a_\alpha = \sum_\beta {g_{ab}{}^\beta}_\alpha e^b_\beta. \tag{1.1}$$

In other words, these are the local components of e^a_α with respect to the frame $\{e^b_\beta\}$. For any section s over $U_a \cap U_b$, denote its local components in U_a and U_b by s^α_a and s^β_b, respectively. Then on $U_a \cap U_b$, we have

$$\begin{aligned} s = \sum_\alpha s^\alpha_a e^a_\alpha &= \sum_{\alpha,\beta} s^\alpha_a \left({g_{ab}{}^\beta}_\alpha e^b_\beta \right) \\ &= \sum_\beta \left(\sum_\alpha {g_{ab}{}^\beta}_\alpha s^\alpha_a \right) e^b_\beta. \end{aligned}$$

Therefore we have the following transition law for local components:

$$s^\beta_b = \sum_\alpha {g_{ab}{}^\beta}_\alpha s^\alpha_a. \tag{1.2}$$

Now, given three charts U_a, U_b, and U_c, it is clear from the definition that these transition functions must satisfy

$$\begin{aligned} g_{aa} &= \mathbf{1} \\ g_{ac} &= g_{bc} \cdot g_{ab} \end{aligned} \tag{1.3}$$

on $U_a \cap U_b \cap U_c$. Here, $\cdot$ denotes matrix multiplication in $\mathrm{GL}(n, K)$. The rule (1.3) is known as the **cocycle condition**.

Conversely, given the data of an open cover $\{U_a\}$ and any collection of $\mathrm{GL}(n, K)$-valued functions g_{ab} on $U_a \cap U_b$, for each a and b, we can construct a vector bundle as follows. Consider the disjoint union of trivial bundles

$$E_0 = \coprod_a U_a \times K^r \tag{1.4}$$

and form the relation

$$(x, v) \in U_a \times K^r \sim (x, g_{ab} \cdot v) \in U_b \times K^r$$

for each a, b. The cocycle condition (1.3) precisely implies that this is an equivalence relation. In particular, we may let

$$E = E_0 / \sim,$$

which has a well-defined projection map to X with fiber K^n. In this way, such gluing data gives rise to a vector bundle over X; the construction is clearly inverse to choosing local frames to form transition functions for a given bundle E.

The notion of isomorphism is very concrete from the perspective of transition functions:

Proposition 1.6. *Two bundles are isomorphic if and only if, after passing to a common refinement $\{U_a\}$, their transition functions are related by*

$$f_{ab} = \tau_a g_{ab} \tau_b^{-1} \tag{1.5}$$

for a collection of $\mathrm{GL}(r, K)$*-valued functions $\{\tau_a\}$ on U_a.*

Proof. Exercise in the definitions. □

Remark 1.7. For the case of line bundles ($r = 1$), we have $\mathrm{GL}(r, K) = K^\times$, so the transition functions and changes-of-frame are just nonvanishing K-valued functions. The cocycle condition (1.3) just says that $\{g_{ab}\}$ is a sheaf cocycle, for the sheaf $\mathscr{C}^0(K^\times)$ of nonvanishing continuous K-valued functions on X, viewed as a sheaf of abelian groups under multiplication. The isomorphism condition (1.5) just says that two cocycles differ by a coboundary. Proposition 1.6 can therefore be rephrased as follows in terms of sheaf cohomology:

Corollary 1.8. *The set of isomorphism classes of line bundles over X is in natural bijection with the sheaf cohomology group $H^1(X, \mathscr{C}^0(K^\times))$.*

A cohomological approach is also possible for higher-rank bundles (in a much more complicated way, called "obstruction theory"), but the homotopy approach that we will take is more powerful (and more standard).

1.6. **Exercises.**

1. Prove that the Möbius bundle is nontrivial.
2. Prove that $TS^2 \oplus \underline{\mathbb{R}}$ is trivial. (See below for definition of direct sum $\oplus$.)
3. Prove Proposition 1.6.

2. Motivating questions, bundle operations (1/27)

2.1. **Two more examples.**

1. The best-known example of a vector bundle is the following. Suppose that $X = M$ happens to be an n-dimensional smooth manifold with a coordinate atlas $\{(U_a, \{x_a^i\}_{i=1}^n\}$. The **tangent bundle** TM of M is the vector bundle with transition functions
$$g_{ab}{}^j{}_i = \frac{\partial x_b^j}{\partial x_a^i}.$$
These are derived from the *coordinate frames* of M, best written symbolically as
$$e_i^a = \frac{\partial}{\partial x_a^i},$$
whence the transition functions above are just the Jacobians of the transition maps between the coordinate charts. The cocycle condition follows from the chain rule. Since the transition functions are smooth $\mathrm{GL}(n, \mathbb{R})$-valued functions on open sets of M, TM is called a **smooth vector bundle**.[3]
2. If M happens to be a complex manifold of complex dimension n, with a holomorphic coordinate atlas $\{(U_a, \{z_a^i\}_{i=1}^n)\}$, we may form the **holomorphic tangent bundle** $T^{1,0}M$ using the transition functions
$$\frac{\partial z_b^j}{\partial z_a^i}.$$
This is a rank n $\mathbb{C}$-vector bundle. Since the transition functions are holomorphic $\mathrm{GL}(n, \mathbb{C})$-valued functions, we call this is a **holomorphic vector bundle**.[4]

2.2. **Motivating questions.** Here are some big motivating questions for the class. The first one is much too big and we will not get to discuss it this semester, although the techniques that we'll introduce turn out to have much to say about it.

Question 1. Which topological manifolds admit smooth structures?

[3]Equivalently, we can define a smooth vector bundle by going through Definition 1.1 and requiring all objects to be smooth manifolds and all projections, sections, etc., to be smooth.

[4]Equivalently, we can define a holomorphic vector bundle by going back through Definition 1.1 and requiring all objects to be complex manifolds and all projections, sections, etc., to be holomorphic...we will do this later.

We will also show that the underlying real vector bundle of $T^{1,0}M$, as a rank $2n$ real vector bundle, is canonically isomorphic to TM, the tangent bundle of the underlying $2n$-dimensional smooth manifold.

Note: a topological manifold is a space that's locally homeomorphic to $\mathbb{R}^n$, *i.e.*, possesses a coordinate atlas with continuous transition maps. Such an atlas is called a *smooth structure* on M if the transition maps are C^∞. (Note: transition *maps* are between coordinate charts on a manifold. Transition *functions* are between local frames in a vector bundle.)

Answer. $n \le 2$, all (we can write them down).

$n = 3$, all (E. Moise, '50s).

$n \ge 4$, most no. In particular:

$n \ge 5$, only finitely many distinct smooth structures per topological manifold.

$n = 4$, still possible that infinitely many non-diffeomorphic smooth structures exist on *every* smooth 4-manifold, as is already known for $\mathbb{R}^4$ and certain closed 4-manifolds.

Smale, Milnor, Freedman, Donaldson, not to mention many non-Fields-medalists... □

The first question that we plan to say something about is in some sense a "linearization" of the above question.

Question 2. Suppose that $X = M$ is a smooth manifold, and $E \to M$ is a topological vector bundle (see Definition 1.1) over M. Can E be given the structure of a smooth vector bundle?

Equivalently: do there exist continuous changes-of-frame $\{\tau_a\}$ on a coordinate atlas $\{U_a\}$ such that the new transition functions $\tau_a g_{ab} \tau_b^{-1}$ are smooth?

Answer. Yes. We should know how to prove this in not too long. □

Question 3. Suppose that M is a complex manifold and $E \to M$ is a smooth (or indeed a topological) complex vector bundle over M. Can E be given the structure of a holomorphic vector bundle?

Equivalently: do there exist smooth changes-of-frame $\{\tau_a\}$ on a coordinate atlas $\{U_a\}$ such that the new transition functions $\tau_a g_{ab} \tau_b^{-1}$ are holomorphic?

Answer. $\dim_{\mathbb{C}} M = 1$, yes.
$\dim_{\mathbb{C}} M \ge 2$, yes, if and only if E carries a connection whose curvature is of type $(1,1)$. We will define these objects in due course. □

Question 4. Suppose that M is a compact Kähler manifold and $E \to M$ is a holomorphic vector bundle. Can E be given the structure of a *flat* bundle, compatible with the holomorphic structure?

Equivalently: do there exist *holomorphic* changes-of-frame $\{\tau_a\}$ on a coordinate atlas $\{U_a\}$ such that the new transition functions $\tau_a g_{ab} \tau_b^{-1}$ are *constant*?

Answer. $\dim_{\mathbb{C}} M = 1$, yes, if and only if the first Chern class vanishes and E is *stable* in Mumford's sense. This is the *Narasimhan-Seshadri Theorem.* We will define both Chern classes and stability in due course.
$\dim_{\mathbb{C}} M \ge 2$, there is an optimal generalization called the *Donaldson-Uhlenbeck-Yau Theorem* which we may get to discuss at the end (for the case $\dim_{\mathbb{C}} M = 2$). □

2.3. **Goal of Part I.** Before trying to make progress on Questions 2-4 above, we have to address the following more basic problem.

Definition 2.1. Given a subgroup $G \subset \mathrm{GL}(n, K)$, we say that a vector bundle has *structure group* G if all of its transition functions take values in G.

A priori, any real vector bundle has structure group $\mathrm{GL}(n, \mathbb{R})$, and any complex vector bundle has structure group $\mathrm{GL}(n, \mathbb{C})$, but one often wants to restrict things further.[5]

Note that having structure group G is just a pointwise condition on the transition functions, so more mundane than the above questions about smooth and holomorphic structures.

The goal of Part I is to give a method to classify all vector bundles with structure group G over a given base space X. We will describe a general answer to this question in terms of homotopy theory, and also show how to extract useful answers in the cases we're interested in.

2.4. **Bundle operations.** We'll now continue developing the bundle formalism.

We have the following **Meta-Theorem.** Any[6] functorial operation on the category of vector spaces gives rise to one on the category of vector bundles.

2.4.1. *Direct sum.* The direct sum of two bundles has total space equal to the fiber product

$$E \oplus F = E \times_X F = \{(v, w) \in E \times F \mid \pi_E(v) = \pi_F(w)\}.$$

The fiber is $E_x \oplus F_x$, and the trivializations are the obvious ones. To check that it is actually a vector bundle, it is easiest to just write down the transition functions. If g is a transition function for E and f is a transition function for F, then the transition function for $E \oplus F$ is

$$g \oplus f,$$

i.e., the induced map on $E_x \oplus F_x$ for each x. By functoriality of $\oplus$ on vector spaces, this preserves the identity and compositions, therefore it preserves the cocycle conditions (1.3).

2.4.2. *Other operations.* Here is a table with all the operations we discussed, together with the transition functions:

Operation	Transition function
$E \oplus F$	$g \oplus f$
$E \otimes_K F$	$g \otimes_K f$
$E^* = \mathrm{Hom}_K(E, K)$	$(g^T)^{-1}$
$\mathrm{End}_K E = E \otimes_K E^*$	$g \otimes (g^T)^{-1}$
$\Lambda^k E$	$k \times k$ minors of g
$\det E$	$\det g$
$\bar{E}$	$\bar{g}.$

Note: For a complex bundle E, $\bar{E}$ is the same underlying real bundle but with a new complex scalar multiplication, defined by

$$\lambda \cdot_{\bar{E}} v := \bar{\lambda} \cdot_E v.$$

[5]There is a formalism (that of *principal bundles*) to make the structure group something more intrinsic, *i.e.*, not just about transition functions with respect to a certain frame. We will introduce this when needed.

[6]This is only true up to a point, as we will discuss below.

2.4.3. *Dual frames and Einstein summation.* One aspect here requires further explanation. Given a frame $\{e_\alpha\}$ for E, we usually work in the *dual frame* $\{e^\alpha\}$ for E^*, which is *uniquely* defined by the rule

$$e^\beta(e_\alpha) = \delta^\beta{}_\alpha.$$

Given sections $s = \sum_\alpha s^\alpha e_\alpha$ of E and $t = \sum_\beta t_\beta e^\beta$ of E^*, we have

$$\begin{aligned} t(s) &= \sum_\beta t_\beta e^\beta \Big(\sum_\alpha s_\alpha e_\alpha\Big) \\ &= \sum_{\alpha,\beta} s^\alpha t_\beta e^\beta(e_\alpha) \\ &= \sum_{\beta,\alpha} s^\alpha t_\beta \delta^\beta{}_\alpha \\ &= \sum_\alpha s^\alpha t_\alpha. \end{aligned}$$

The **Einstein summation convention** says that when pairing sections of E and its dual bundle E^*, *using the dual frame*, we can omit the $\sum_\alpha$ and just write

$$t(s) = s^\alpha t_\alpha.$$

So, when used properly, the convention should always be about summing the local components of a section of a bundle (up index) and its dual bundle (down index) in the dual frame. The mathematical content of the convention is that this is an *a priori* well-defined scalar.

Of course, it is common to abuse the notation, for instance to omit the Σ_α when writing $s = s^\alpha e_\alpha$ to define the local components themselves. But one should be aware that when used properly, the Einstein convention has mathematical content.

Example 2.2. In the special case that M is a smooth manifold of dimension n and $E = TM$, and we are working in a coordinate chart $\{x^i\}_{i=1}^n$, the tangent bundle has a distinguished *coordinate frame*

$$\left\{e_i = \frac{\partial}{\partial x^i}\right\}$$

as described above in §2.1. The dual frame of the cotangent bundle $E^* = T^*M$ is written as:

$$\{e^i = dx^i\},$$

and satisfies

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i{}_j.$$

Ordinarily, **we will use the Latin indices** i, j, k, ℓ, **only for a coordinate frame/coframe of the tangent bundle.** So, Greek indices refer to an arbitrary frame for a vector bundle E, but Latin indices refer to a specific type of frame for the bundles TM and T^*M over a smooth manifold.[7]

[7]The letters a, b, c that we have attached to coordinate charts above are *labels*, not *indices*. Eventually we will omit these labels completely...an abuse of notation that is critical to our ability to do differential geometry.

2.4.4. *Endomorphisms.* Following this convention, given a frame for E, we also have a natural frame for the endomorphism bundle

$$\mathrm{End}_K E = \mathrm{Hom}_K(E,E) = E \otimes_K E^*$$

given by

$$\{e_\beta \otimes e^\alpha\}.$$

So a section of the endomorphism bundle

$$\sigma = \sum_{\alpha,\beta} \sigma^\beta{}_\alpha e_\beta \otimes e^\alpha$$

acts on a section $s = \sum_\gamma s^\gamma e_\gamma$ of E by

$$\begin{aligned} \sigma(s) &= \left(\sum_{\alpha,\beta} \sigma^\beta{}_\alpha e_\beta \otimes e^\alpha\right)\left(\sum_\gamma s^\gamma e_\gamma\right) \\ &= \sum_{\alpha,\beta,\gamma} \sigma^\beta{}_\alpha s^\gamma \delta^\alpha{}_\gamma e_\beta \\ &= \sum_{\alpha,\beta} (\sigma^\beta{}_\alpha s^\alpha) e_\beta. \end{aligned}$$

Using Einstein summation, we can just write

$$(\sigma(s))^\beta = \sigma^\beta{}_\alpha s^\alpha.$$

So we wind up with the usual matrix multiplication rule.

We shall use similar notations on all tensor products between bundles and their duals, so that we can always pair an up and a down index of the same bundle.

2.5. **Subbundles.** A **subbundle** $E \subset F$ is by definition a subset that has the structure of a vector bundle over X in the subspace topology. We have the following:

Lemma 2.3. *For an injective bundle morphism $\varphi : E \to F$, the image $\varphi(E) \subset F$ is a subbundle, and $\varphi : E \to \varphi(E)$ is an isomorphism.*

Proof. It suffices to show that the map $\varphi : E \to \varphi(E)$ is a homeomorphism, where $\varphi(E) \subset F$ is given the subspace topology.

Since the question is local, we may let $x_0 \in X$ and choose a coordinate neighborhood U on which we identify

$$E|_U \cong U \times K^n.$$

Let $v_\alpha = \varphi_{x_0}(e_\alpha) \in K^n$, for $\alpha = 1, \ldots, k$. By assumption, these are linearly independent, so we may complete them to a basis

$$\{v_1, \ldots, v_k, v_{k+1}, \ldots v_n\}$$

for K^n. Let $V = \langle v_{k+1}, \ldots, v_n\rangle \subset K^n$. Then we may define a bundle morphism over U by

$$\begin{aligned} \varphi \oplus \mathbf{1} : E|_U \oplus \underline{V} &\to F|_U \\ (s,v) &\mapsto \varphi(s) + v. \end{aligned}$$

By assumption, this is an isomorphism on the fiber E_{x_0}. By the usual argument (see proof of Proposition 1.3), this implies that it is a bundle isomorphism over a neighborhood $W \ni x_0$.

The map φ over W is obtained by restricting this isomorphism to two subspaces that are in bijection; therefore, φ is also a homeomorphism over W. Since $x_0 \in X$ was arbitrary, we are done. □

2.6. **Exercises.**

1. Check that the transition functions of E^* with respect to dual frames are given by $(g^T)^{-1}$.
2. Convince yourself that
$$\Lambda^k(E \oplus F) \cong \oplus_{i+j=k} \Lambda^i E \otimes \Lambda^j F.$$
3. Show that the direct sum or tensor product of two copies of the Möbius bundle is trivial.
4. Show that for any line bundle, $L \otimes L^*$ is canonically isomorphic to the trivial bundle. (*Hint:* write down an obvious morphism to the trivial line bundle.)

3. Quotients, metrics, splitting (2/1)

3.1. **Quotients and exact sequences.** In the proof of Lemma 2.3, we established the following fact: given a subbundle $E \subset F$, it is always possible to choose local frames for F of the form

$$\underbrace{\{\overbrace{e_1,\dots,e_k}^{\text{frame for } E}, e_{k+1},\dots,e_n\}}_{\text{frame for } F}.$$

With respect to frames of this form, the transition functions of f look like

$$f_{ab} = \begin{pmatrix} g_{ab} & \ell_{ab} \\ 0 & k_{ab} \end{pmatrix}. \tag{3.1}$$

We have

$$\begin{aligned} f_{bc} f_{ab} &= \begin{pmatrix} g_{bc} g_{ab} & g_{bc}\ell_{ab} + \ell_{bc} k_{ab} \\ 0 & k_{bc} k_{ab} \end{pmatrix} \\ &= f_{ac} = \begin{pmatrix} g_{ac} & \ell_{ac} \\ 0 & k_{ac} \end{pmatrix}. \end{aligned} \tag{3.2}$$

From the lower-right block, we conclude that

$$k_{bc} k_{ab} = k_{ac},$$

so the matrices $\{k_{ab}\}$ satisfy the cocycle condition. We may therefore define the **quotient bundle** F/E to be the bundle with transition functions $\{k_{ab}\}$.

From this construction, the fibers of the quotient bundle come with identifications

$$(F/E)_x \cong F_x / E_x.$$

The quotient bundle therefore has the universal property with respect to bundle morphisms expressed by the following diagram:

$$\begin{tikzcd} E \arrow[r] \arrow[dr, "0"'] & F \arrow[r] \arrow[d] & F/E, \arrow[dl, dashed, "\exists!"'] \\ & G & \end{tikzcd} \tag{3.3}$$

where one only needs to check using local frames that the induced map is continuous (exercise). Owing to this universal property, the quotient we have constructed is uniquely defined up to isomorphism.

In a very similar fashion, we can define the **kernel** of any surjective bundle morphism $F \to G$ *of constant rank*.

A **short exact sequence** of vector bundles is a diagram

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$$

such that the induced maps on fibers are exact sequences of vector spaces (necessarily of constant rank). In particular, any exact sequence implies isomorphisms $G \cong F/E$ and $E \cong \ker(F \to G)$.

Here is a useful fact about short exact sequences of vector bundles:

Lemma 3.1. *An exact sequence of vector bundles*

$$0 \to E \to F \overset{f}{\to} G \to 0$$

induces a canonical isomorphism

$$\det F \cong \det E \otimes \det G. \tag{3.4}$$

Proof. Write $k = \mathrm{rk} E$, $r = \mathrm{rk} F$, and let U be a sufficiently small open set containing a given point. Over U, we can choose a lift u such that $f \circ u = \mathbf{1}_G$. Define a map

$$\det E \otimes \det G \to \det F$$
$$v_1 \wedge \cdots \wedge v_k \otimes w_1 \wedge \cdots \wedge w_{r-k} \mapsto v_1 \wedge \cdots \wedge v_k \wedge u(w_1) \wedge \cdots \wedge u(w_{r-k}).$$

Suppose we change to another lift in which $\tilde{u}(w_i)$ is replaced by $u(w_i) + v_i'$, for $v_i' \in \Gamma(U, E)$. Then the above element maps to

$$\begin{aligned} v_1 \wedge \cdots \wedge v_k \wedge & (u(w_1) + v_1') \wedge \cdots \wedge (u(w_{r-k}) + v_{r-k}') \\ &= v_1 \wedge \cdots \wedge v_k \wedge u(w_1) \wedge (u(w_2) + v_2') \wedge \cdots \wedge (u(w_{r-k}) + v_{r-k}') \\ &\quad + v_1 \wedge \cdots \wedge v_k \wedge v_1' \wedge (u(w_2) + v_2') \wedge \cdots \wedge (u(w_{r-k}) + v_{r-k}') \\ &= v_1 \wedge \cdots \wedge v_k \wedge u(w_1) \wedge (u(w_2) + v_2') \wedge \cdots \wedge (u(w_{r-k}) + v_{r-k}'), \end{aligned}$$

since $v_1 \wedge \cdots \wedge v_k \wedge v_1' = 0$. Continuing, we have

$$\begin{aligned} v_1 \wedge \cdots \wedge v_k \wedge & u(w_1) \wedge (u(w_2) + v_2') \wedge \cdots \wedge (u(w_{r-k}) + v_{r-k}') \\ &= v_1 \wedge \cdots \wedge v_k \wedge u(w_1) \wedge (u(w_2) + v_2') \wedge \cdots \wedge (u(w_{r-k}) + v_{r-k}') \\ &= v_1 \wedge \cdots \wedge v_k \wedge u(w_1) \wedge u(w_2) \wedge \cdots \wedge (u(w_{r-k}) + v_{r-k}') \\ &= \cdots = v_1 \wedge \cdots \wedge v_k \wedge u(w_1) \wedge \cdots \wedge u(w_{r-k}). \end{aligned}$$

Hence, the map is independent of the choice of local lift, so gives a globally well-defined bundle morphism. Since it is clearly a local isomorphism, it is a global isomorphism (3.4).

Alternatively, one can obtain the isomorphism at the level of transition functions, directly from (3.1). ☐

3.2. **The tautological bundle over $\mathbb{CP}^1$.** Recall

$$\begin{aligned}\mathbb{CP}^1 &= \{\text{complex lines through the origin in } \mathbb{C}^2\}\\ &= \{[u,v] \in \mathbb{C}^2 \setminus \{(0,0)\}\}/[u,v] \sim [\lambda u, \lambda v]\,\forall \lambda \in \mathbb{C}^\times.\end{aligned} \tag{3.5}$$

This is covered by the two coordinate charts

$$U_0 = \{[1,z] \mid z \in \mathbb{C}\}, \qquad U_1 = \{[w,1] \mid w \in \mathbb{C}\}.$$

The **tautological bundle** over $\mathbb{CP}^1$ is the subbundle of $\underline{\mathbb{C}}^2 = \mathbb{CP}^2 \times \mathbb{C}^2$ defined by

$$\mathscr{O}(-1) = \{([u,v],(a,b)) \mid av - bu = 0\}.$$

The fiber over $x = [u,v]$ is the plane

$$\mathscr{O}(-1)_x = \{(a,b) \in x\} = \{(\lambda u, \lambda v) \mid \lambda \in \mathbb{C}\}.$$

We define the **hyperplane bundle** $\mathscr{O}(1)$ over $\mathbb{CP}^1$ by the exact sequence

$$0 \to \mathscr{O}(-1) \to \underline{\mathbb{C}}^2 \to \mathscr{O}(1) \to 0. \tag{3.6}$$

(In other words, we define $\mathscr{O}(1) = \underline{\mathbb{C}}^2/\mathscr{O}(-1)$ for the above inclusion.) According to Lemma 3.1, we have an isomorphism

$$\mathscr{O}(-1) \otimes \mathscr{O}(1) \cong \Lambda^2\underline{\mathbb{C}}^2 \cong \underline{\mathbb{C}}.$$

Tensoring with $\mathscr{O}(-1)^*$, and using the fact $\mathscr{O}(-1)^* \otimes \mathscr{O}(-1) \cong \underline{\mathbb{C}}$ (Exercise 2.6.4), we obtain

$$\mathscr{O}(1) \cong \underline{\mathbb{C}} \otimes \mathscr{O}(1) \cong \mathscr{O}(-1)^* \otimes \mathscr{O}(-1) \otimes \mathscr{O}(1) \cong \mathscr{O}(-1)^* \otimes \underline{\mathbb{C}} \cong \mathscr{O}(-1)^*. \tag{3.7}$$

We obtain

$$\mathscr{O}(1) = \mathscr{O}(-1)^*,$$

which is an alternate definition.

An equivalent way of describing this isomorphism is to fix a nondegenerate skew-symmetric $\mathbb{C}$-bilinear form on $\mathbb{C}^2$, for instance

$$\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

For any subbundle $\mathscr{L} \subset \underline{\mathbb{C}}^2$ (indeed over any base), this sets up an isomorphism

$$\begin{aligned}\mathscr{L} &\xrightarrow{\sim} \left(\underline{\mathbb{C}}^2/\mathscr{L}\right)^*\\ v &\mapsto \omega(v,\cdot).\end{aligned} \tag{3.8}$$

3.3. Metrics and splitting of exact sequences.

Definition 3.2. A **metric** $\langle \cdot, \cdot \rangle$ on a vector bundle E is a continuously varying inner product on the fibers. For $K = \mathbb{R}$, this means a real inner product

$$K = \mathbb{R} : \langle v, w \rangle_x = \langle w, v \rangle_x \in \mathbb{R},$$

and for $K = \mathbb{C}$, this means a Hermitian inner product. We take a convention that a Hermitian inner product is complex-linear in the second coordinate, so we have

$$K = \mathbb{C} : \langle \lambda v, w \rangle = \overline{\langle w, \lambda v \rangle} = \overline{\lambda \langle w, v \rangle} = \bar{\lambda} \overline{\langle w, v \rangle} = \bar{\lambda} \langle v, w \rangle \in \mathbb{C}.$$

More precisely, a real inner product is a global section of $E^* \otimes_{\mathbb{R}} E^*$, and a Hermitian inner product is a global section of $\bar{E}^* \otimes_{\mathbb{C}} E^*$.

Proposition 3.3. *Every vector bundle over a paracompact base space carries a metric.*

Note: A topological space is called **paracompact** if it is Hausdorff and every open cover has a locally finite subcover. By an argument involving Urysohn's Lemma, this is equivalent to the statement that every open cover admits partitions of unity!

Proof. We abuse notation somewhat.

Given a system of local trivializations $\{U_a\}$, choose a subordinate partition of unity $\{\rho_a\}$. Let $\delta^a_{\alpha\beta}$ be the Euclidean inner product in the trivialization U_α. Then $\rho_a \delta^a_{\alpha\beta}$ (no sum) is a well-defined global section of $\bar{E}^* \otimes E^*$. Let

$$\langle \cdot, \cdot \rangle = \sum_a \rho_a \delta^a_{\alpha\beta}.$$

On each fiber E_x, this is a convex linear combination of inner products, so is itself an inner product. □

Corollary 3.4. *In the category of (topological) vector bundles, every exact sequence splits.*

Proof. Let $E \subset F$ be a subbundle, and choose a metric on F. Define $G = E^\perp \subset F$. This is locally the kernel of a surjective map from F to K^{n-k}, so is a subbundle, and clearly $E \oplus G = F$, whence $G \cong F/E$. □

Corollary 3.5. *For any complex bundle, we have $\bar{E} \cong E^*$.*

Proof. The isomorphism is given by $v \mapsto \langle \cdot, v \rangle$, which is complex-conjugate-linear as a map from E, hence linear as a map from $\bar{E}$. □

Corollary 3.6. *Any real vector bundle admits a reduction of structure group to* $\mathrm{O}(n)$. *Any complex vector bundle admits a reduction of structure group to* $\mathrm{U}(n)$. *The reductions are unique up to isomorphism of* $\mathrm{O}(n)$ *(or* $\mathrm{U}(n)$*)-bundles.*

Proof. Choose a metric on E. Given any local frame $\{e_\alpha\}$, the Gram-Schmidt process uniquely reduces $\{e_\alpha\}$ to an orthonormal frame:

$$\begin{aligned} \{e_1, e_2 \ldots, e_n\} &\rightsquigarrow \{e_1, \tilde{e}_2, e_3, \ldots, e_n\} \\ &\rightsquigarrow \{e_1, \tilde{e}_2, \tilde{e}_3, e_4, \ldots, e_n\} \\ &\rightsquigarrow \cdots \rightsquigarrow \\ &\rightsquigarrow \{e_1, \tilde{e}_2, \tilde{e}_3, \ldots, \tilde{e}_n\}. \end{aligned} \tag{3.9}$$

This amounts to right-multiplying by an upper-triangular matrix with positive real entries on the diagonal. Letting ${g^\beta}_\alpha$ be a transition function between two such orthonormal frames $\{e_\alpha\}$ and $\{f_\alpha\}$, we have

$$\begin{aligned}
\delta_{\alpha\beta} &= \langle e_\alpha, e_\beta \rangle \\
&= \left\langle {g^\gamma}_\alpha f_\gamma, {g^\delta}_\beta f_\delta \right\rangle \\
&= {\bar{g}^\gamma}_\alpha {g^\delta}_\beta \langle f_\gamma, f_\delta \rangle \\
&= {\bar{g}^\gamma}_\alpha {g^\delta}_\beta \delta_{\gamma\delta}.
\end{aligned} \tag{3.10}$$

In other words, we have

$$\mathbf{1} = \bar{g}^T g,$$

so g belongs to $\mathrm{U}(n)$, as claimed.

The uniqueness part requires a bit more thought. It should amount to Gram-Schmidt again plus the fact that the group of upper-triangular matrices with positive diagonals intersects $\mathrm{U}(n)$ in the identity. □

3.4. **The category of vector bundles.** Here we summarize some of the good and bad properties of the category of vector bundles over X.

1. The tensor product $\otimes$ is an **exact functor**, because the same is true for vector spaces. (For general modules, $\otimes$ or only right-exact.)
2. Exact sequences split. This is not true holomorphically, though.
3. The global sections functor

$$\Gamma(\cdot) = \Gamma(X, \cdot)$$

is **exact**. Left-exactness is true for sheaves generally. On topological/smooth bundles, it is also right-exact, because exact sequences split. Again, this is not true for holomorphic bundles.
4. There is a natural map

$$\Gamma(E) \times \Gamma(F) \to \Gamma(E \otimes F),$$

but this may not be surjective. This is actually a key feature of the subject because it allows you to "generate" new sections by tensoring with powers of a certain kind of line bundle.
5. The category of vector bundles is **not an abelian category**, because kernels and cokernels only exist (as vector bundles) for morphisms of *constant rank*. In the complex-analytic or algebraic setting, the category of *coherent sheaves* is the smallest abelian category that contains vector bundles, which is a very manageable sub-category of the sheaves of $\mathscr{O}_X$-modules. In the smooth category (*i.e.* the category of sheaves of C^∞_X-modules), quotients are not necessarily finitely generated modules, so what you get is a big mess as far as I know.

3.5. Exercises.

1. Check that the map in the "∃!" arrow in the universal property of the quotient (3.3) is a bundle morphism.
2. Define the tautological bundle $\mathscr{O}_{\mathbb{RP}^1}(-1)$ over $\mathbb{RP}^1$ in a similar way as for $\mathbb{CP}^1$. Check that $\mathscr{O}_{\mathbb{RP}^1}(-1)$ is the Möbius bundle. (Here we identify $\mathbb{RP}^1$ with S^1 by the map $\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} \mapsto \begin{pmatrix}\cos 2\theta\\ \sin 2\theta\end{pmatrix}$.)
3. Check that $\Gamma(\cdot)$ is exact on the category of vector bundles over X. (*I.e.* topological or smooth vector bundles).
4. Give an example of a real line bundle L such that the natural map
$$\Gamma(L) \times \Gamma(L) \to \Gamma(L^2)$$
is not surjective.

4. Pullbacks, homotopy theorem (2/3)

4.1. **Pullbacks and bundle maps.** The last bundle operation we'll discuss is of fundamental importance.

Definition 4.1. Suppose that $E \to Y$ is a vector bundle and $f : X \to Y$ is any map. We may form the **pullback** f^*E as follows: the total space is the fiber product
$$f^*E = E \times_Y X = \{(v, x) \mid \pi(v) = f(x)\},$$
and addition is defined by the same rule as on E. If $(U, \{e_\alpha\})$ is a local trivialization for E, then
$$(f^{-1}(U), \{e_\alpha \circ f\})$$
is a local trivialization for f^*E, as one can check; the transition functions of f^*E are obtained by precomposing the transition functions of E with f.

The pullback has the following obvious properties:

1. $\mathbf{1}^*E = E$
2. $g^*f^*E \cong (f \circ g)^*E$
3. $f^*(E \oplus F) \cong f^*E \oplus f^*F$
4. $f^*(E \otimes F) \cong f^*E \otimes f^*F$.

These properties can be summarized by saying that *pullback by $f : X \to Y$ induces a natural transformation from the category of vector bundles on Y to vector bundles on X.*

The definition of pullback comes in tandem with the following one.

Definition 4.2. Suppose that $E' \to X$ and $E \to Y$ are vector bundles of the same rank, r. A map $\varphi : E' \to E$ is called a *bundle map* if it carries each vector space E'_x isomorphically onto some E_y.

In particular, a bundle map induces a continuous function $\varphi_0 : X \to Y$ by sending $x \in X$ to $\pi_E(\varphi(E_x))$. In other words, we have the following diagram:

$$\begin{tikzcd} E' \arrow[r, "\varphi"] \arrow[d] & E \arrow[d] \\ X \arrow[r, "\varphi_0"] & Y \end{tikzcd}$$

IMPORTANT WARNING: A bundle *morphism* is not always a bundle *map*, because the latter is required to be an isomorphism on fibers. However, note that in the case $X = Y$ and $\varphi_0 = \mathbf{1}$, a bundle map is exactly a bundle *iso*morphism.

The concepts of pullback and bundle map turn out to be completely equivalent.

Lemma 4.3. *Suppose that* $\varphi : E' \to E$ *is any bundle map. Then we have a canonical isomorphism*

$$E' \cong \varphi_0^* E.$$

Proof. The existence of a map $E' \to \varphi_0^* E$ follows from the universal property of the fiber product:

$$\begin{tikzcd} E' \arrow[dr, dashed] \arrow[drr, bend left, "\varphi"] \arrow[ddr, bend right] & & \\ & \varphi_0^* E \arrow[r] \arrow[d] & E \arrow[d] \\ & X \arrow[r, "\varphi_0"] & Y. \end{tikzcd}$$

This is clearly a bundle morphism, and has maximal rank since φ does, so is an isomorphism. □

4.2. **Examples.**

1. The pullback by a constant map $f : X \to y_0 \in Y$ is a trivial bundle $X \times E_{y_0}$.
2. Given a subspace $S \subset X$ and $\iota : S \to X$ the inclusion map, the pullback of a bundle E is simply the **restriction**

$$\iota^* E = E|_S \, ,$$

   which has total space $\pi^{-1}(S)$ and trivializations obtained by intersecting open sets of X with S.
3. Let $p : M \to N$ be any *covering map* between smooth manifolds, *i.e.*, the differential $dp_x : T_x M \to T_{p(x)} N$ is an isomorphism for each x. (The fibers are discrete.) Then dp is a bundle map, so according to the Lemma, we have

$$p^* TN \cong TM.$$

4. Let $p : S^n \to \mathbb{RP}^n$ be the projection map, whose fibers consist of pairs of antipodal points. By the previous item, we have an isomorphism $p^* T\mathbb{RP}^n \cong TS^n$. Meanwhile, the pullback $p^* \mathscr{O}_{\mathbb{RP}^n}(-1)$ is isomorphic to the trivial bundle over S^n (exercise).

5. Let $\pi : E \to X$ be the projection map for a vector bundle. We can restrict this map to the open set $E \smallsetminus \underline{0}$, where $\underline{0}$ is the zero section, and consider the pullback
$$\pi^* E \to E \smallsetminus \underline{0}.$$
The resulting vector bundle has a "tautological" section, given by
$$(x, v) \mapsto (v, (x, v)).$$
In fact this is a nonvanishing global section, so spans a trivial 1-dimensional subbundle of $\pi^* E$. Iterating the construction, one can eventually trivialize the pullback of E completely.[8]

4.3. **The Homotopy Theorem.** What makes the pullback construction so important is the following property.

Theorem 4.4. *Suppose that $f_0, f_1 : X \to Y$ are homotopic. Then for any bundle $E \to Y$,*
$$f_0^* E \cong f_1^* E.$$

Recall that two maps f_0, f_1 are *homotopic* if there exists $f_t : X \times [0,1] \to Y$ such that $f_{t=0} = f_0$ and $f_{t=1} = f_1$. Given such f_t, we can pull back $E \to Y$ to obtain a bundle
$$f_t^* E \to X \times [0,1]$$
for which $f_t^* E|_{X \times \{0\}} \cong f_0^* E$ and $f_t^* E|_{X \times \{1\}} \cong f_1^* E$ (since restriction is a form of pullback, and pullbacks are transitive). Theorem 4.4 therefore follows directly from:

Theorem 4.5 (Homotopy Theorem)**.** *Suppose $E \to X \times [0,1]$ is any vector bundle over a paracompact space X, and write $E_t = E|_{X \times \{t\}}$. Then*
$$E_0 \cong E_1.$$

We will prove the special case that X is *compact* (and Hausdorff). For a slick proof in the paracompact case, see Hatcher VBKT, Theorem 1.6.

Lemma 4.6. *Let E and F be vector bundles over $X \times [0,1]$, with X compact, and suppose that $E_t \cong F_t$. Then there exists $\varepsilon > 0$ such that*
$$E|_{X \times (t-\varepsilon, t+\varepsilon)} \cong F|_{X \times (t-\varepsilon, t+\varepsilon)}.$$

Proof. A morphism $E \to F$ is equivalent to a section of $\mathrm{Hom}(E, F) = F \otimes E^*$. This is an isomorphism if and only if the determinant of the local component matrix with respect to any local frames at any point.

By assumption, we have a section σ_0 of $\mathrm{Hom}(E_t, F_t) = \mathrm{Hom}(E, F)|_{X \times \{t\}}$. We can choose a *finite* (since X is compact) open cover of X, $\{U_a\}$, with the property that both E and F are trivial over $\{U_a \times (t - \varepsilon_a, t + \varepsilon_a)\}$, for each a. (This is because products form a subbasis for the topology on $X \times [0,1]$.)

[8] This construction is usually applied not to $E \smallsetminus \underline{0}$ but to its quotient under scalar multiplication, the *projectivization* $\mathbb{P}(E)$. This is a fiber bundle with fiber $\mathbb{P}(E_x)$ and structure group $PGL(n, K)$. Rather than a nonvanishing section, one obtains a tautological subbundle of the pullback. After finitely many iterations, one obtains a decomposition of E into a direct sum of line bundles. This is known as the *splitting principle*, and is an important trick in Bott and Tu's book for example.

Now, since E and F are trivial over $U_a \times (t-\varepsilon_a, t+\varepsilon_a)$, $\sigma_0|_{U_a}$ extends trivially to a section

$$\sigma_0^a \in \Gamma(U_a \times (t-\varepsilon_a, t+\varepsilon_a), \mathrm{Hom}(E,F)).$$

Let $\{\rho_a\}$ be a partition of unity subordinate to $\{U_a\}$. Then the sections

$$\rho_a \sigma_0^a \in \Gamma(X \times (t-\varepsilon_a, t+\varepsilon_a), \mathrm{Hom}(E,F))$$

are well-defined and continuous.

Let $\varepsilon_0 = \min_a \varepsilon_a > 0$. We may form the sum

$$\sigma = \sum_a \rho_a \sigma_0^a \in \Gamma(X \times (t-\varepsilon_0, t+\varepsilon_0), \mathrm{Hom}(E,F)).$$

Since $\rho_a \sigma^a|_{U_a \times \{t\}} = \rho_a \sigma$, we have

$$\sigma|_{X\times\{t\}} = \sum_a \rho_a \sigma_0^a|_{X\times\{t\}} = \sum_a \rho_a \sigma_0 = \sigma_0.$$

Therefore σ restricts to the isomorphism σ on $X \times \{t\}$. By the usual argument, the local determinants are nonvanishing in a neighborhood of $X \times \{t\}$, and σ is an isomorphism. Since X is compact, we may take this neighborhood of the form $X \times (t-\varepsilon, t+\varepsilon)$ for some $\varepsilon > 0$. □

Proof of Theorem 4.5. Let $p_t : X \times [0,1] \to X \times \{t\}$ be the projection map. Define $F^t = p_t^* E$. By definition, we have

$$E_t \cong (F^t)_t\,.$$

By the lemma, for each $t \in [0,1]$, there exists $\varepsilon_t > 0$ such that

$$E|_{X\times\{t-\varepsilon_t, t+\varepsilon_t\}} \cong F^t|_{X\times(t-\varepsilon_t, t+\varepsilon_t)}\,.$$

But clearly $(F^t)_s$ are all isomorphic to E_t, so we conclude that

$$E_s \cong E_t$$

for all $t - \varepsilon_t \le s \le t + \varepsilon_t$.

Now, choose a finite cover of $[0,1]$ by open intervals of the form $(t-\varepsilon_t, t+\varepsilon_t)$. Choose a finite set of times

$$0 = t_0, t_1, t_2, \ldots, t_N = 1$$

such that $[t_i, t_{i+1}]$ is contained in one of these intervals, for each i. We then have

$$E_0 = E_{t_0} \cong E_{t_1} \cong E_{t_2} \cong \cdots \cong E_{t_N} = E_1,$$

as desired. □

We write $\mathrm{Vect}_{n,K}(X)$ for the set of isomorphism classes of vector bundles of rank n over K.

Corollary 4.7. *A homotopy equivalence $f : X \to Y$ induces a natural bijection*

$$f^* : \mathrm{Vect}_{n,K}(Y) \xrightarrow{\sim} \mathrm{Vect}_{n,K}(X).$$

Proof. Let g be a homotopy inverse to f, so $f \circ g \sim \mathbf{1} \sim g \circ f$. Then

$$(f \circ g)^* = g^* f^* = \mathbf{1}^* = \mathbf{1}$$

by Property 1 following Definition 4.1. Similarly, $f^* g^* = \mathbf{1}$, so f^* is a bijection. □

Corollary 4.8. *Any bundle E over a contractible space X is trivial.*

Proof. Let $f_t : X \times [0,1] \to X$ be the contracting map, with $f_1 = \mathbf{1}$ and $f_0(X) = x_0 \in X$. By the homotopy theorem, we have

$$f_1^* E = E \cong f_0^* E = X \times E_{x_0}$$

by Example 4.2.1. □

4.4. **Exercises.**

1. Justify the statement in Example 4.2.4.: the pullback of the tautological bundle $\mathscr{O}_{\mathbb{RP}^n}(-1) \to \mathbb{RP}^n$ by the projection map $p : S^n \to \mathbb{RP}^n$ is trivial.
2. Modify the proof of the Homotopy Theorem to show that in fact $E \cong p_0^* E_0$ over $X \times [0,1]$.
3. Read the proof of the paracompact case of the Homotopy Theorem in Hatcher VBKT, Theorem 1.6; or try to devise a proof without looking it up.

5. Structure group, orientation (2/8)

5.1. **Determinant bundles and reduction to** $\mathrm{SO}(n)$ **and** $\mathrm{SU}(n)$**.** Fix a group $G \subset \mathrm{GL}(n, K)$, and recall that E is said to have *structure group* G if all its transition functions take values pointwise in G. We say that two bundles with structure group G are **G-isomorphic** (often just **isomorphic**) if there exists an isomorphism such that the $\{\tau_a\}$ in (1.5) also take values in G.[9]

For any subgroup $H \subset \mathrm{G}$, note that any H-bundle is in particular a G-bundle. Given a G-bundle E, we say that the structure group of E can be **reduced** to H if there is an H-bundle E' such that $E' \cong E$ as G-bundles. This is equivalent to choosing a new collection of G-equivalent local frames for E such that the transition functions with respect to these frames belong to H.

Recall from the proof of Corollary 3.6 that choosing a metric on E has the effect of reducing the structure group of any rank r line bundle to $\mathrm{O}(r)$, in the real case, or $\mathrm{U}(r)$, in the complex case. In geometry, it's equally important to fix an orientation (when possible) as well as a metric on a given manifold. This is a question of choosing a volume form, *i.e.*, a section of the top exterior power of the cotangent bundle. The same notion is important for general vector bundles, and so we make the following

Definition 5.1. A real vector bundle is called **orientable** if $\det E$ has a nonvanishing global section, or equivalently, is a trivial line bundle. (The equivalence follows from Corollary 1.5.)

We now want to give an equivalent formulation in terms of reduction of structure groups. The relevant group is $\mathrm{SO}(n)$, which we reintroduce; for future use, we'll also calculate its 0'th and 1st homotopy groups.

[9]We will at some point introduce the intrinsic definition of G-bundles, where the definition of isomorphism can be made without reference to local frames.

Recall that $\mathrm{SO}(n) \subset \mathrm{O}(n)$ is the subgroup with determinant $+1$, which is an index-2 subgroup. We have $\mathrm{SO}(1) \cong \langle e \rangle$, and there is fibration

$$\mathrm{SO}(n) \to \mathrm{SO}(n+1) \to S^n \tag{5.1}$$

for each n, so it follows by induction that $\mathrm{SO}(n)$ is connected for all n. Therefore $\mathrm{O}(n)$ has two connected components.

We have $\mathrm{SO}(2) \cong \mathrm{U}(1) \cong S^1$, so $\pi_1(\mathrm{SO}(2)) \cong \mathbb{Z}$. To compute the fundamental groups of $\mathrm{SO}(3)$ and higher, recall that

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \mid z, w \in \mathbb{C}, |z|^2 + |w|^2 = 1 \right\}. \tag{5.2}$$

Clearly $\mathrm{SU}(2) \cong S^3 \subset \mathbb{C}^2$. Now, the adjoint action of $\mathrm{SU}(2)$ on its Lie algebra, the real 3-dimensional space of skew-Hermitian 2×2 complex matrices, is an isometric action with stabilizer $\pm\mathbf{1}$. This can be checked explicitly from the definition (5.2) (exercise). The action therefore sets up a fibration

$$\mathbb{Z}_2 \to \mathrm{SU}(2) \to \mathrm{SO}(3).$$

The long exact sequence of this fibration implies that $\pi_1(\mathrm{SO}(3)) \cong \pi_0(\mathbb{Z}_2) \cong \mathbb{Z}_2$. Then, the fibration (5.1) implies that $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}_2$ for all $n \geq 3$. To summarize, we have

$$\pi_1(\mathrm{SO}(n)) \cong \begin{cases} \mathbb{Z} & n = 2 \\ \mathbb{Z}_2 & n \geq 3. \end{cases}$$

The relevance of $\mathrm{SO}(n)$ here is as follows.

Proposition 5.2. *A real vector bundle is orientable iff its structure group reduces to* $\mathrm{SO}(n)$.

Proof. For the ($\Leftarrow$) direction, given E with structure group $\mathrm{SO}(n)$, the transition functions of $\det E$ are by definition equal to one. Hence $\det E$ is a trivial bundle.

For the ($\Rightarrow$) direction, suppose given both a nonvanishing section V of $\det E$ and a metric h on E. By choosing orthonormal frames as in Corollary 3.6, we can reduce the structure group of E to $\mathrm{O}(n)$. The metric h also induces a metric on $\det E$, and we let

$$\tilde{V} = \frac{V}{|V|_h}$$

to obtain a volume form of norm one. Now, reorder each local frame (if necessary) so that $\tilde{V} = e_1 \wedge \cdots \wedge e_r$. Then for a transition function g between two such local frames e_α and e'_α, we have

$$\tilde{V} = e_1 \wedge \cdots \wedge e_r = (\det g)\, e'_1 \wedge \cdots \wedge e'_r = \det g\, \tilde{V}. \tag{5.3}$$

So $\tilde{V} = \det g\, \tilde{V}$, and $\det g = 1$, so g takes values in $\mathrm{SO}(n)$. ∎

Next, we discuss the complex case. The unitary groups each fit into a fibration

$$\mathrm{U}(n) \to \mathrm{U}(n+1) \to S^{2n+1}. \tag{5.4}$$

Since $\mathrm{U}(1) \cong S^1$, we have $\pi_0(\mathrm{U}(1)) = 0$ and $\pi_1(\mathrm{U}(1)) = \mathbb{Z}$, and the long exact sequence of a fibration gives

$$\pi_0(\mathrm{U}(n)) = \langle e \rangle, \qquad \pi_1(\mathrm{U}(n)) = \mathbb{Z} \tag{5.5}$$

for all $n \geq 1$. Since $\mathrm{U}(n)$ is connected, we must have $\mathrm{U}(n) \subset \mathrm{SO}(2n)$. Therefore:

Corollary 5.3. *The underlying real bundle of any complex vector bundle is orientable.*

Recall that $\mathrm{SU}(n) \subset \mathrm{U}(n)$ is the subgroup with *complex* determinant one. So we have a fibration

$$\mathrm{SU}(n) \to \mathrm{U}(n) \overset{\det}{\to} U(1). \tag{5.6}$$

This fibration / exact sequence of groups has a section:

$$\mathrm{U}(1) \to \mathrm{U}(n)$$
$$e^{i\theta} \mapsto \begin{pmatrix} e^{i\theta} & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

From this we learn that topologically, $\mathrm{U}(n) \cong \mathrm{SU}(n) \times \mathrm{U}(1)$ (although as groups it is a semidirect product), so $\mathrm{SU}(n)$ is connected and simply connected for all n.

Proposition 5.4. *For a complex vector bundle E of rank r, the complex determinant bundle $\det_{\mathbb{C}} E = \Lambda^r E$ is trivial iff the structure group reduces to* $\mathrm{SU}(n)$.

Proof. The ($\Leftarrow$) direction is the same as in Proposition 1.6. For the ($\Rightarrow$) direction, we can as above reduce to $\mathrm{U}(n)$ and choose a complex volume form $\tilde{V}$ of norm one. Now, in each local frame, we have $\tilde{V} = f(x) e_1 \wedge \cdots \wedge e_r$, where $f(x)$ is complex-valued of norm one. Just replace e_1 by $e_1 / f(x)$. By the same argument as in (5.3), we conclude that the transition functions have complex determinant one, so belong to $\mathrm{SU}(n)$.[10] ☐

5.2. **Some more remarks.** We should point out the most famous feature of these homotopy groups. Notice that from the exact sequences of the fibrations (5.1) and (5.4), it follows that

$$\pi_i(\mathrm{O}(n)) \cong \pi_i(\mathrm{O}(n+1)) \text{ for } n > i + 1$$

and

$$\pi_i(\mathrm{U}(n)) \cong \pi_i(\mathrm{U}(n+1)) \text{ for } n > i/2.$$

For each i, we can therefore define the *stable homotopy groups*

$$\pi_i^s(\mathrm{O}(n)) = \lim_{n \to \infty} \pi_i(\mathrm{O}(n)),$$

and

$$\pi_i^s(\mathrm{U}(n)) = \lim_{n \to \infty} \pi_i(\mathrm{U}(n)).$$

Bott discovered that these are periodic in i, with period 8 and 2, respectively. In particular, for $\mathrm{U}(n)$, the formula (5.5) describes all the stable homotopy groups for even and odd i.

Bott periodicity was originally proved by applying Morse theory to the loop spaces of these Lie groups, but can also be arrived at via K-theory of real or complex vector bundles, which involves only the tools we're developing in this part of the course. For a lucid account of this glorious story, I recommend Bott's article *The periodicity theorem for the classical groups and some of its applications* (1970). Details of the K-theory proof of the periodicity theorem appear in the books by Atiyah and Hatcher listed in the references.

[10] Thanks to Anuk for pointing out that a previous version of this argument was stupid.

The Morse theory ideology behind Bott's original proof also fed back into the theory of vector bundles through Atiyah and Bott's paper *The Yang-Mills equations on a Riemann surface* (1983) and in the development of Floer theory, still perhaps the hottest research area in geometry/topology. Another recommended survey article by Bott, touching on these developments, is *Morse Theory Indomitable* (1988).

5.3. **Exercises.**

1. Check the above claim that $\mathrm{SU}(2)$ acting by conjugation on its Lie algebra of 2×2 skew-Hermitian matrices sets up an isomorphism
$$\mathrm{SU}(2)/\langle \pm\mathbf{1}\rangle \xrightarrow{\sim} \mathrm{SO}(3).$$
(Hint: To begin with, how does the U(1)-subgroup
$$\left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \mid \theta \in S^1 \right\}$$
act on $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$? How does it act on the subspace
$$\left\{ \begin{pmatrix} 0 & -\bar{w} \\ w & 0 \end{pmatrix} \mid w \in \mathbb{C} \right\}?)$$
2. Justify the cancellation property used in the two proofs above: if s is a nonvanishing section of a line bundle and $s = f\cdot s$ for a function f, then $f \equiv 1$.

6. Clutching construction, quaternionic line bundles (2/8-10)

6.1. **The clutching construction.** We are now in a position to classify vector bundles with a given structure group over the n-sphere.

The classification rests on the following construction. The sphere S^n is a union of two closed hemispheres $S^n_\pm$, both of which are homeomorphic to disks centered at the north and south poles, $p_\pm$. Their boundaries coincide in an equatorial sphere of one lower dimension:
$$\partial S^n_\pm = \pm S^{n-1}.$$
We also have
$$N_\varepsilon(S^n_+) \cap N_\varepsilon(S^n_-) = N_\varepsilon(S^{n-1}),$$
where N_ε means open ε-neighborhood. Since $N_\varepsilon(S^n_\pm)$ deformation retracts onto $S^n_\pm$, preserving S^{n-1}, it is completely equivalent to construct bundles using the closed cover $S^n = S^n_+ \cup S^n_-$ instead of the open cover $S^n = N_\varepsilon(S^n_+) \cup N_\varepsilon(S^n_-)$.

Given any map $g : S^{n-1} \to G$, we can simply construct the bundle with transition function g between S^n_+ and S^n_-. We refer to this bundle as E_g. This is referred to as the **clutching construction.**

Examples 6.1. 1. The Möbius bundle over S^1 is associated to the clutching function

$$g : S^0 \to \mathrm{O}(1) \cong S^0 \subset \mathbb{R}^\times$$
$$\pm 1 \mapsto \pm 1. \tag{6.1}$$

2. Consider the tangent bundle $TS^2 \to S^2$. Drawing the frames coming down/up from the north/south pole to the equator, one can see that

$$\begin{pmatrix} e_1^+ \\ e_2^+ \end{pmatrix} = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} e_1^- \\ e_2^- \end{pmatrix}.$$

This matrix is g_{+-}, in the convention (1.1), and corresponds to rotation by 2θ when acting on $\mathbb{R}^2$ in the usual way.

3. For the tautological bundle $\mathscr{O}(-1) \to \mathbb{CP}^1 \cong S^2$, we have two frames:

$$[1, z] \mapsto (1, z) \in \mathbb{C}^2$$

over U_0, and

$$[w, 1] \mapsto (w, 1) \in \mathbb{C}^2$$

over U_1. On $U_0 \cap U_1$, we have $w = z^{-1}$, and

$$(1, z) = z(z^{-1}, 1) = z(w, 1).$$

So our clutching function is

$$z = e^{i\theta} : S^1 \to \mathrm{U}(1) \cong S^1.$$

Let's now find the transition function of the underlying real rank 2 bundle. We have

$$e_1^+ = (1, z), \quad e_2^+ = i(1, z), \quad e_1^- = (w, 1), \quad e_2^- = i(w, 1).$$

So, since $z = \cos\theta + i \sin\theta$, our transition function goes:

$$e_1^+ = (1, z) = z(w, 1) = \cos\theta e_1^- + \sin\theta e_2^-$$

$$e_2^+ = i(1, z) = iz(w, 1) = -\sin\theta e_1^- + \cos\theta e_2^-.$$

Therefore, the transition matrix g_{+-} corresponds to rotation by $-\theta$ on $\mathbb{R}^2$.

4. We define the bundle $\mathscr{O}(n) \to S^2 \cong \mathbb{CP}^1$ to be the complex line bundle with clutching function z^{-n}. For $n \in \mathbb{Z}$, we conclude:

$$\mathscr{O}(n) \cong \mathscr{O}(1)^{\otimes n}. \tag{6.2}$$

From the previous example, we find:

$$TS^2 \cong \mathscr{O}(2). \tag{6.3}$$

Here is the main result about this construction. We write $[S^k, G]$ for the set of unbased homotopy classes of maps $S^k \to G$. (Note that this is a surjective image of the based homotopy group $\pi_k(G)$ if G is connected, and they are the same if G is simply connected.)

Theorem 6.2. *Every bundle over S^n arises from the clutching construction. If $[g_0] = [g_1] \in [S^{n-1}, G]$, then E_{g_0} and E_{g_1} are isomorphic over S^n. Conversely, assuming G is connected, if $E_f \cong E_g$ for two functions $f, g : S^{n-1} \to G$, then $[f] = [g] \in [S^{n-1}, G]$.*

In summary, we have

$$\left[S^{n-1}, G\right] \cong \{G\text{-bundles over } S^n\}/G\text{-isomorphism}$$

assuming G is connected; if not, we have a surjective map.

Proof. For the first statement, given any bundle $E \to S^n$, according to Corollary 4.8, the restrictions to the contractible sets $S^n_\pm$ are trivial. Therefore E can be trivialized over these two sets, with a transition function $g : S^{n-1} \to G$. Hence $E \cong E_g$.

For the homotopic ⇒ isomorphic direction, we prove the case $G = \mathrm{GL}(r, K)$ and leave the case of general G as an exercise. Let $g_t : S^{n-1} \times [0,1] \to \mathrm{GL}(r, K)$ be the homotopy between g_0 and g_1. Consider the two sets $S^n_+ \times [0,1]$ and $S^n_- \times [0,1]$, which cover $S^n \times [0,1]$ and intersect along $S^{n-1} \times [0,1]$. We can use g_t as a clutching function over $S^{n-1} \times [0,1]$, to obtain a bundle

$$E \to S^n \times [0,1]$$

for which $E|_{S^n \times \{i\}} \cong E_{g_i}$, for $i = 0, 1$. By the homotopy theorem, we must have

$$E_{g_0} \cong E_{g_1},$$

as claimed.

For the isomorphic ⇒ homotopic direction, let $E, F \to S^n$. Given trivializations of E and F over $S^n_\pm$, we have transition functions $g : S^{n-1} \to G$ for E and $f : S^{n-1} \to \mathrm{G}$ for F.

Suppose $\tau : E \to F$ is an isomorphism over S^n, and write

$$\tau^\pm = \tau|_{S^n_\pm}$$

for the G-valued functions giving the isomorphism. These are related by

$$f \cdot \tau^+ = \tau^- \cdot g$$

on S^{n-1}, or

$$(\tau^-)^{-1} \cdot f \cdot \tau^+ = g.$$

We have $S^{n-1} = \partial S^n_+ = \partial D^n$. Then we can simply write

$$\tau^+_{tx} = \tau^+|_{S^{n-1}_t},$$

where S^{n-1}_t is the sphere at angle $t\pi/2$ from the north pole, p_+. We have

$$\tau^+_{tx} = \begin{cases} \tau^+ & t = 1 \\ \tau^+(p_+) & t = 0 \end{cases}.$$

In particular, $\tau^+ \sim \tau(p_+)$ is homotopic to a constant map. We therefore have

$$g = (\tau^-)^{-1} f \tau^+ \sim \left(\tau^-(p_-)\right)^{-1} f \tau^+(p_+).$$

Since G is connected by assumption, we further have

$$\tau^\pm(p_\pm) \sim \mathbf{1}.$$

Therefore

$$g \sim \left(\tau^-(p_-)\right)^{-1} f \tau^+(p_+) \sim f,$$

as desired. □

Corollary 6.3. *Every complex line bundle over S^2 is isomorphic to $\mathscr{O}(n)$, for some n, and these are non-isomorphic for distinct n. Hence,*

$$\mathrm{Vect}_{1,\mathbb{C}}(S^2) \cong \mathbb{Z}.$$

Proof. By Corollary 3.6, any complex line bundle has a reduction of structure group to $\mathrm{U}(1) \cong S^1$, which is connected. By the previous theorem, we have

$$\mathrm{Vect}_{1,\mathbb{C}}(S^2) \cong \left[S^1, \mathrm{U}(1)\right] \cong \mathbb{Z}.$$

□

Corollary 6.4 (Hairy ball theorem)**.** *The tangent bundle of S^2 has no nonvanishing global sections.*

Proof. We saw in Example 6.1 above that $TS^2 \cong \mathscr{O}(2)$ as a complex line bundle. By the previous corollary, this is nontrivial, so cannot have a nonvanishing global section. □

Corollary 6.5. *Every real line bundle over S^n, $n \geq 2$, is trivial. Every complex line bundle over S^n, $n \geq 3$, is trivial.*

Proof. This is because $\pi_i(S^k) = 0$ for $k = 0$ and $i \geq 1$, or for $k = 1$ and $i \geq 2$. □

Corollary 6.6. *Under the clutching construction, the set of isomorphism classes of* $\mathrm{SU}(2)$*-bundles over S^4 corresponds to $\mathbb{Z}$.*

Proof. By the discussion in the previous subsection, we have $\mathrm{SU}(2) \cong S^3$, so $\pi_3(\mathrm{SU}(2)) \cong \mathbb{Z}$. □

Corollary 6.7. $\mathrm{Vect}_{2,\mathbb{C}}(S^4) \cong \mathbb{Z}$.

Proof. Let $E \to S^4$ be a rank 2 complex vector bundle. We know from Corollary 6.5 that $\det E$ is trivial, so the structure group reduces to $\mathrm{SU}(2)$. The result follows by the previous corollary.

This can also be proved by calculating $\pi_3(U(2))$ directly from the fibration (5.6). □

6.2. $\mathrm{SU}(2)$**-bundles and quaternionic line bundles.** We saw above that there is a $\mathbb{Z}$'s worth of $\mathrm{SU}(2)$-bundles over S^4. It is worth describing a generator of these, for future use.

First, we should discuss $\mathrm{SU}(2)$-bundles generally, which is best done using the quaternions. Recall that $\mathbb{H}$ is the 4D real associative algebra generated by

$$q_i, i = 0, 1, 2, 3$$

where[11]

$$q_0 = \mathbf{1}, q_i^2 = -\mathbf{1} \text{ for } i = 1, 2, 3, \text{ and } q_1 q_2 q_3 = \mathbf{1}.$$

[11] These quaternions are given in terms of Hamilton's quaternions by $q_1 = -\mathbf{i}, q_2 = -\mathbf{j}, q_3 = -\mathbf{k}$. We use this convention in order that the standard action of $\mathrm{SU}(2)$ will be given by quaternion multiplication on the left.

The quaternions act on themselves both by left- and by right-multiplication, and it turns out that these actions together give all of $\operatorname{End}\mathbb{R}^4$. This is part of why the quaternions are so useful.

For a quaternion

$$x = x^0 q_0 + x^1 q_1 + x^2 q_2 + x^3 q_3,$$

we have the conjugation map

$$\bar{x} = x^0 q_0 - x^1 q_1 - x^2 q_2 - x^3 q_3,$$

which satisfies $\overline{xy} = \bar{y}\bar{x}$. A quaternion has real and imaginary parts

$$\operatorname{Re} x = \frac{1}{2}(x + \bar{x}), \quad \operatorname{Im} x = \frac{1}{2}(x - \bar{x}).$$

We have a standard quaternionic inner product

$$\langle x, y \rangle = \bar{x} y \in \mathbb{H} \tag{6.4}$$

whose real part corresponds to the standard inner product on $\mathbb{R}^4$. In particular, for x as above, we have

$$|x|^2 = \bar{x}x = \sum_{i=0}^{4} (x_i)^2 = x\bar{x}. \tag{6.5}$$

The inverse of a nonzero quaternion is therefore given by

$$x^{-1} = \frac{\bar{x}}{|x|^2}.$$

This makes $\mathbb{H}$ into a normed associative division algebra (one of only three).

We also want to think about $\mathbb{H}$ as a complex vector space. For this, choose the complex structure on $\mathbb{H}$ given by *right*-multiplication by q_1:

$$ix = x \cdot q_1.$$

With this complex structure, we get to make the standard identification of $\mathbb{H} \cong \mathbb{R}^4$ with $\mathbb{C}^2$:

$$\begin{gathered} \mathbb{C}^2 \xrightarrow{\sim} \mathbb{H} \\ \begin{pmatrix} x^0 + ix^1 \\ x^2 + ix^3 \end{pmatrix} \mapsto q_0(x^0 + x^1 q_1) + q_2(x^2 + x^3 q_1). \end{gathered} \tag{6.6}$$

Since it is an associative algebra, the action of $\mathbb{H}$ by *left*-multiplication on $\mathbb{H}$ automatically commutes with the complex structure. Moreover, the action by unit-norm quaternions preserves the norm $|\cdot|$ (exercise). Therefore, left-multiplication by the unit quaternions $S^3 \subset \mathbb{H}$ automatically belongs to the group $\mathrm{U}(2) = \mathrm{GL}(2, \mathbb{C}) \cap \mathrm{O}(4)$. Since this is a 3-dimensional subgroup, it must be $\mathrm{SU}(2)$. More explicitly, one can check (exercise) that the left-action in the above basis on $\mathbb{C}^2$ is given by:

$$q_1 \cdot - = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$
$$q_2 \cdot - = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{6.7}$$
$$q_3 \cdot - = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

In particular, our identification agrees with (5.2).

Conversely, $\mathrm{SU}(2)$ is exactly the subgroup of $\mathrm{O}(4)$ (or of $SL(4,\mathbb{R})$) consisting of elements that commute with the right-action of $\mathbb{H}$ on itself.[12] The right-action also gives a copy of $\mathrm{SU}(2)$, so there is a map

$$\mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SO}(4)$$

whose kernel is $\pm(\mathbf{1}, \mathbf{1})$. Hence, the LHS is the universal cover of the $\mathrm{SO}(4)$, also known as $\mathrm{Spin}(4)$:

$$\mathrm{SU}(2) \times \mathrm{SU}(2) = \mathrm{Spin}(4).$$

Proposition 6.8. *Every complex rank 2 bundle with structure group* $\mathrm{SU}(2)$ *determines a quaternionic line bundle, and vice-versa.*[13]

Proof. ($\Rightarrow$) Since $\mathrm{SU}(2)$ commutes with the right-action of $\mathbb{H}$ on $\mathbb{H} \cong \mathbb{C}^2$ in any local trivilization (using the above identification), this gives a globally well-defined scalar multiplication by $\mathbb{H}$ (on the right).

($\Leftarrow$) To reduce the structure group, choose a global quaternionic metric generalizing (6.4), *i.e.*, a map

$$\langle \cdot, \cdot \rangle : E \times E \to \mathbb{H}$$

which is right-$\mathbb{H}$-linear in the second entry and satisfies[14]

$$\overline{\langle v, w \rangle} = \langle w, v \rangle .$$

This can be done as in the real and complex cases. One can use the same Gram-Schmidt procedure to make the frames quaternionically orthonormal. Then the transition functions will preserve the real part of the inner product and commute with the right $\mathbb{H}$-action, so belong to $\mathrm{SU}(2)$.[15] □

[12] On a quaternionic vector space of dimension n, this group is called $\mathrm{Sp}(n)$, or sometimes called the *compact symplectic group*. So we have the exceptional isomorphism $\mathrm{SU}(2) \cong \mathrm{Sp}(1)$. Take care not to confuse $\mathrm{Spin}(n)$, the connected double cover of $\mathrm{SO}(n)$ for $n \geq 3$, with $\mathrm{Sp}(n)$.

[13] We shall always take quaternionic vector spaces to have a scalar multiplication by $\mathbb{H}$ on the *right*.

[14] Maybe this can be stated in terms of quaternionic tensor products of quaternionic line bundles with their bars, but I don't want to go there.

[15] More generally, this argument lets you reduce the structure group of a rank-n quaternionic vector bundle to $\mathrm{Sp}(n)$.

Example 6.9. Let

$$\begin{aligned} \mathbb{HP}^1 &= \{\text{quaternionic lines through the origin in } \mathbb{H}^2\} \\ &= \{[u,v] \in \mathbb{H}^2 \smallsetminus \{(0,0)\}\}/[u,v] \sim [u\lambda, v\lambda]\ \forall \lambda \in \mathbb{H}^\times. \end{aligned} \tag{6.8}$$

This is covered by the two coordinate charts

$$U_0 = \{[1,z] \mid z \in \mathbb{H}\}, \qquad U_1 = \{[w,1] \mid w \in \mathbb{H}\}.$$

As with the complex numbers, it follows that

$$\mathbb{HP}^1 \cong S^4.$$

The **tautological bundle** over $\mathbb{HP}^1$ is the subbundle of $\underline{\mathbb{H}}^2 = \mathbb{HP}^1 \times \mathbb{H}^2$ defined by

$$\mathscr{O}_{\mathbb{HP}^1}(-1) = \{([u,v], (a,b)) \mid (a,b) \in [u,v]\}.$$

The fiber over $[u,v]$ is the plane

$$\{(u\lambda, v\lambda) \mid \lambda \in \mathbb{H}\}.$$

As in Example 6.1.3., one can compute the transition function

$$g_{+-}(x) = x \cdot -$$

for $x \in S^3 \cong \mathrm{SU}(2)$. So under the clutching construction, this bundle corresponds to $1 \in \pi_3(\mathrm{SU}(2)) \cong \mathbb{Z}$.

6.3. **Exercises.**

1. Give a proof of the (homotopic $\Rightarrow$ isomorphic) direction in Theorem 6.2 for general G. (Hint: you can do it directly from (1.5) without referencing the homotopy theorem.)
2. Prove that every complex line bundle over S^1 is trivial.
3. Prove that every real vector bundle of rank 2 over S^n, $n \geq 3$, is trivial.
4. Show that $\mathscr{O}(2) \oplus \underline{\mathbb{C}} \cong \mathscr{O}(1) \oplus \mathscr{O}(1)$ over S^2. (For a short argument, see Hatcher VBKT, Example 1.13.)
5. Show that the action of right- or left-multiplication by unit-norm quaternions preserves the norm (6.5).
6. Check (6.7).

7. Universal vector bundles, characteristic classes (2/15)

Definition 7.1. A rank r vector bundle $E_r \to Y$ over K is said to be **universal** if every rank r vector bundle $E \to X$ is the pullback $E = f^*E_r$ by some map $f : X \to Y$, unique up to homotopy.

If it exists, a universal bundle is unique up to homotopy equivalence: suppose that $E'_r \to Y'$ is another universal bundle. Then $E'_r = f^*E_r$ for some $f : Y' \to Y$ and $E_r = g^*E'_r$ for some $g : Y \to Y'$. Hence, $g^*f^*E_r = E_r$, so $g^*f^* \sim \mathbf{1}$ by uniqueness, and $f^*g^* \sim 1$ by symmetrical argument.

The space Y (up to homotopy equivalence) is called the **classifying space** for $\mathrm{Vect}_{r,K}$.

7.1. **Projective spaces.** Let $K = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, and let $K\mathbb{P}^n$ be the projective space of lines in K^{n+1}. $K\mathbb{P}^n$ is covered by charts

$$U_i = \{[X^0 : \dots : X^n] : X^i \neq 0\} \xrightarrow{\cong} K^n, \quad [X^0 : \dots : X^n] \mapsto (X^0/X^i, \dots, \widehat{X^i/X^i}, \dots, X^n/X^i).$$

There are inclusions

$$K\mathbb{P}^n \hookrightarrow K\mathbb{P}^{n+1}, \quad [X^0 : \dots : X^n] \mapsto [X^0 : \dots : X^n : 0].$$

This direct limit of this system is $K\mathbb{P}^\infty$, where $S \subset K\mathbb{P}^\infty$ is closed if and only if $S \cap K\mathbb{P}^n$ is closed for all n.

The **tautological bundle** of $K\mathbb{P}^n$ is

$$\mathscr{O}_{K\mathbb{P}^n}(-1) = \{(x, v) \in K\mathbb{P}^n \times K^{n+1} : v \in x\}.$$

If we think of $\underline{K}^{n+1}$ as a sub-bundle of $\underline{K}^{n+2}$, then we obtain inclusions $\mathscr{O}_{K\mathbb{P}^n}(-1) \hookrightarrow \mathscr{O}_{K\mathbb{P}^{n+1}}(-1)$. The direct limit of this system is $\mathscr{O}_{K\mathbb{P}^\infty}(-1) \subset \underline{K}^\infty_X$.

Note: K^∞ is the infinite direct sum, consisting of tuples where all but finitely many entries are zero. This is not the same thing as Hilbert space!

By reading Hatcher's *Algebraic Topology*, or knowing that cup product is dual to intersection product, one gets

Proposition 7.2. *For $n \leq \infty$, one has:*

- $H^*(\mathbb{R}\mathbb{P}^n, \mathbb{Z}_2) \cong \mathbb{Z}_2[x]/x^{n+1}$, *where* $\deg x = 1$
- $H^*(\mathbb{C}\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}[x]/x^{n+1}$, *where* $\deg x = 2$
- $H^*(\mathbb{H}\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}[x]/x^{n+1}$, *where* $\deg x = 4$.

The inclusions $K\mathbb{P}^n \to K\mathbb{P}^{n+1}$ induce the obvious maps on cohomology. Taking the inverse limit yields

$$H^*(K\mathbb{P}^\infty, R) \cong R[x],$$

where $R = \mathbb{Z}_2$ when $K = \mathbb{R}$ and $R = \mathbb{Z}$ otherwise.

Let $k = \dim_{\mathbb{R}} K$. The unit sphere bundle inside

$$\mathscr{O}(-1) \hookrightarrow \underline{K}^{n+1} \to K\mathbb{P}^n$$

is

$$S^{k-1} \to S^{k(n+1)-1} \to K\mathbb{P}^n.$$

Letting $n \to \infty$, we obtain

$$S^{k-1} \to S^\infty \to K\mathbb{P}^\infty.$$

Since S^∞ is contractible, the LES for homotopy groups implies

$$\pi_i(K\mathbb{P}^\infty) \cong \pi_{i-1}(S^{k-1}).$$

Specializing K, we obtain

- $K = \mathbb{R}$ then $\mathbb{R}\mathbb{P}^\infty \cong K(\mathbb{Z}_2, 1)$
- $K = \mathbb{C}$ then $\mathbb{C}\mathbb{P}^\infty \cong K(\mathbb{Z}, 2)$.

7.2. **Grassmannians.** Let $K = \mathbb{R}$ or $\mathbb{C}$. The **Grassmannian** of r-planes in K^n is

$$G_{r,n} = \{r\text{-planes in } K^n\}.$$

To topologize this set, fix a metric on K^n and embed $G_{r,n}$ via

$$\begin{aligned} G_{r,n} &\to K^{n^2} \\ X &\mapsto \pi_X \in \operatorname{End}(K^n) \cong K^n \otimes (K^n)^* \cong K^{n^2}, \end{aligned}$$

where π_X is the orthogonal projection onto X. The Grassmannian inherits the subspace topology.

The Grassmannian is covered by $\binom{n}{r}$ charts, corresponding to the $r \times r$ minors of an $r \times n$ projection matrix π_X. Each chart is diffeomorphic to $K^{r(n-r)}$ because $r(n-r)$ is the number of free entries in the row-reduced echelon form of $r \times n$ matrix of rank r.

There are inclusions $G_{r,n} \to G_{r,n+1}$, with limit $G_r = \{r\text{-planes in } K^\infty\}$. The **tautological bundle** on G_r is

$$E_r = \{(x,v) : v \in x\} \subset \underline{K}^\infty.$$

An interesting fact that we may prove later:

- if $K = \mathbb{R}$, then $H^*(G_r) \cong \mathbb{Z}_2[x_1, \dots, x_r]$ with $\deg x_i = i$ and
- if $K = \mathbb{C}$, then $H^*(G_r) \cong \mathbb{Z}[x_1, \dots, x_r]$ with $\deg x_i = 2i$.

7.3. **Tautological bundles on Grassmannians are universal.**

Theorem 7.3. *$E_r \to G_r$ is universal for rank r bundles on paracompact bases when $K = \mathbb{R}, \mathbb{C}$ (or $K = \mathbb{H}$ and $r = 1$.)*

Lemma 7.4. *Let X be a paracompact space. Bundle maps*

$$\begin{tikzcd} E \arrow[r] \arrow[d] & E_r \arrow[d] \\ X \arrow[r] & G_r \end{tikzcd}$$

are equivalent to injective morphisms

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d] & \underline{K}^\infty_X \arrow[dl] \\ X. & \end{tikzcd}$$

Proof. Define the map tautologically and check continuity locally. □

Proof of Theorem 7.3. Choose a locally finite cover $\{U_a\}$ on which E is trivialized. Let ρ_a be a partition of unity subordinate to the cover. Since the cover is trivializing, there are bundle maps $\varphi_a : E|_{U_a} \to U_a \times K^r \to K^r$.

Since $\operatorname{supp} \rho_a \subset U_a$, we obtain a map $\rho_a \varphi_a : E \to K^r$, which is an injective bundle morphism on $\operatorname{supp}(\rho_a)$. Define

$$\varphi = \bigoplus_a \rho_a \varphi_a : E \to \bigoplus_a (K^r) = K^\infty$$

(wlog, $\{U_a\}$ is countable). This morphism is injective, so by Lemma 7.4, we obtain a bundle *map*

$$\begin{array}{ccc} E & \longrightarrow & E_r \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & G_r. \end{array}$$

To finish, we check this is unique up to homotopy. Suppose $f, g : X \to G_r$ are such that

$$f^* E_r \cong E \cong g^* E_r.$$

We must show that $f \sim g$. Equivalently, given $\varphi, \psi : E \to \underline{K}_X^\infty$ injective, we must show that φ and ψ are homotopic through injective morphisms.

Define a homotopy from $\alpha_0 = \mathbf{1}$ to

$$\alpha_1(x_1, x_2, \ldots) = (x_1, 0, x_2, 0, x_3, 0, \ldots)$$

by

$$\alpha_t : K^\infty \to K^\infty, \quad (x_1, x_2, \ldots) \mapsto (x_1, (1-t)x_2, (1-t)x_3 + tx_2, \cdots).$$

Similarly, define β_t such that $\beta_0 = \mathbf{1}$ and

$$\beta_1(x_1, x_2, \ldots) = (0, x_1, 0, x_2, 0, \ldots).$$

The homotopy

$$F_t = (1-t)\alpha_1 \circ \varphi + t\beta_1 \circ \psi$$

is injective for all t. This gives us

$$\varphi = \alpha_0 \circ \varphi \sim \alpha_1 \circ \varphi = F_0 \sim F_1 = \beta_1 \circ \psi \sim \beta_0 \circ \psi = \psi.$$

□

7.4. Characteristic classes.

Definition 7.5. A **characteristic class** for rank r vector bundles is a natural transformation

$$c : \mathrm{Vect}_{r,K}(-) \to H^*(-, R),$$

for some ring R.

In other words, for all bundles $E \to X$, we obtain a class $c(E) \in H^*(X, R)$, which must satisfy

$$c(f^*E) = f^*(c(E)) \tag{7.1}$$

for all $f : Z \to X$.

Lemma 7.6. *Characteristic classes c correspond bijectively to cohomology classes $\bar{c} \in H^*(G_r)$.*

Proof. ($\to$) Given a characteristic class c, define $\bar{c} = c(E_r)$.

($\leftarrow$) Given $\bar{c} \in H^*(G_r)$, define

$$c(E) = f_E^* \bar{c},$$

where $f_E : X \to G_r$ is a classifying map for E. Since classifying maps are unique up to homotopy, $c(E)$ is well-defined. This is functorial because given a map $g : Z \to X$, we obtain a diagram

$$\begin{tikzcd} g^*E \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & E_r \arrow[d] \\ Z \arrow[r, "g"] & X \arrow[r, "f"] & G_r. \end{tikzcd}$$

Since $f \circ g$ is covered by a bundle map, it is a classifying map for g^*E. This gives us

$$c(g^*E) = (f \circ g)^*E = g^*f^*E = g^*c(E)).$$

So $c(-)$ satisfies (7.1) and is indeed a characteristic class. $\square$

Definition 7.7. Let L be a line bundle ($r = 1$) over $K = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. Notice that $G_1 = K\mathbb{P}^\infty$. So by (7.2), we have the following characteristic classes:

- $K = \mathbb{R}$, $w_1(L)$ corresponding to $x \in \mathbb{Z}_2[x] \cong H^*(\mathbb{RP}^\infty, \mathbb{Z}_2)$, with $\deg x = 1$, is the first **Stiefel-Whitney class**.
- $K = \mathbb{C}$, $c_1(L)$ corresponding to $x \in \mathbb{Z}[x] \cong H^*(\mathbb{CP}^\infty, \mathbb{Z})$, with $\deg x = 2$, is the first **Chern class.**
  Note: by convention, we choose the generator x such that $c_1(\mathscr{O}(-1)) = -x$.
- $K = \mathbb{H}$, $\tilde{p}_1(L)$ corresponding to $x \in \mathbb{Z}[x] \cong H^*(\mathbb{HP}^\infty, \mathbb{Z})$, with $\deg x = 4$, is the first **Pontryagin class**.[16]

7.5. Exercises.

1. Think through the proof of Lemma 7.4.
2. * Think about the infinite quaternionic Grassmannian for $r > 1$. Is it a classifying space for bundles over $\mathbb{H}$ of rank $r > 1$?
3. * Try to develop the theory of infinite-rank bundles, either modeled on K^∞ or on Hilbert space. Can you make a universal bundle?
4. * Read about K-theory and the classifying spaces BO and BU.

8. Classification of vector bundles in low dimension, transversality (2/17)

8.1. All topological bundles on a smooth manifold are smooth.

Let's finally get Question 1 from Section 2.2 out of the way.

Theorem 8.1. *Let M be a smooth manifold and $E \to M$ a vector bundle. Then there exists a unique smooth bundle $E' \to M$ such that $E \cong_{top} E'$.*

Proof. Let $f : M \to G_r = \lim_{n\to\infty} G_{r,n}$ be the classifying map of E. By cellular approximation, there is a cellular map

$$f_1 : M \to G_{r,N}$$

[16] Warning: the characteristic class $\tilde{p}_1(L)$ that we have just defined for an SU(2)-bundle is actually $\pm\frac{1}{2}$ times the ordinary Pontryagin class of the underlying real vector bundle.

(for some large N) with $f_1 \sim f$. The restriction

$$E_r|_{G_{r,N}} \to G_{r,N}$$

is smooth by construction. By smooth approximation, there is $f_2 \sim f_1$ such that

$$f_2 : M \to G_{r,N}$$

is a smooth map. Now take

$$E' = f_2^* E_r|_{G_{r,N}} \cong E.$$

□

Theorem 8.2. *If two smooth bundles are topologically isomorphic, then they are smoothly isomorphic.*

Proof. This follows by uniqueness in the last construction. Alternatively, one can simply take the continuous section of $\mathrm{Hom}(E, E') = E' \otimes E^*$ giving the isomorphism and approximate it by a smooth section. □

8.2. **Classification of line bundles.** Recall the first examples of characteristic classes given in Definition 7.7 above. In the case of line bundles ($r = 1$) over $\mathbb{R}$ or $\mathbb{C}$, these classes actually tell us everything that's going on.

Theorem 8.3. *Real and complex line bundles are classified up to isomorphism by w_1 and c_1, respectively.*

Proof. We have

$$\begin{aligned} \{\text{real line bundles on } X\} &\cong [X, \mathbb{RP}^\infty] \\ &\cong [X, K(\mathbb{Z}_2, 1)] \\ &\cong H^1(X, \mathbb{Z}_2). \end{aligned}$$

The same proof works for complex bundles, replacing $\mathbb{RP}^\infty$ with $\mathbb{CP}^\infty = K(\mathbb{Z}, 2)$. □

Another proof of Theorem 8.3 for Chern classes. The SES of groups

$$0 \to \mathbb{Z} \overset{2\pi i \cdot}{\to} \mathbb{C} \overset{\exp(\cdot)}{\to} \mathbb{C}^* \to 0$$

gives rise to a short exact sequence of sheaves

$$0 \to \underline{\mathbb{Z}} \to C^0_X(\mathbb{C}) \to (C^0_X(\mathbb{C}))^\times \to 0.$$

The LES of this sequence is

$$\cdots \to H^1(C^0_X(\mathbb{C})) \to H^1(C^0_X(\mathbb{C})^\times) \overset{c_1'}{\to} H^2(X, \mathbb{Z}) \to H^2(C^0_X(\mathbb{C})) \to \cdots.$$

As X is paracompact, $C^0_X(\mathbb{C})$ is a fine sheaf; therefore

$$H^1(C^0_X(\mathbb{C})) = H^2(C^0_X(\mathbb{C})) = 0$$

and c_1' is an isomorphism. The LES in cohomology is functorial under pullback of sheaves, so c_1' is a characteristic class.

Since the pullback on cohomology induced by $\mathbb{CP}^1 \to \mathbb{CP}^\infty$ is an isomorphism in degree 2, to show that $c_1 = c_1'$ it suffices to check that $c_1'(\mathscr{O}_{\mathbb{CP}^1}(-1)) = -[\mathbb{CP}^1]$. This is surprisingly annoying. We will check this later when we have the best definition of Chern classes. □

Another important fact about w_1 and c_1 is:

Lemma 8.4. *Let L, L' be K-line bundles.*

- *If $K = \mathbb{R}$, then $w_1(L \otimes_{\mathbb{R}} L') = w_1(L) + w_1(L')$.*
- *If $K = \mathbb{C}$, then $c_1(L \otimes_{\mathbb{C}} L') = c_1(L) + c_1(L')$.*

Proof. We do the complex case; the real case is the same.

The classifying map for $\mathscr{O}_{\mathbb{CP}^1}(n)$ is

$$\mathbb{CP}^1 \overset{z \mapsto z^{-n}}{\to} \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty.$$

We know this based on clutching functions. This is a degree $-n$ map, so the pullback of the generator of $H^*(\mathbb{CP}^1)$ is $-n[\mathbb{CP}^1]$. This shows that

$$c_1(\mathscr{O}(1)^{\otimes n}) = c_1(\mathscr{O}(n)) = nc_1(\mathscr{O}(1)),$$

which gives us the desired statement for line bundles over $\mathbb{CP}^1$. This also gives the statement for the bundles $\mathscr{O}(n)$ over $\mathbb{CP}^\infty$.

Here is how you can get the general statement (which I admit I completely failed to prove in class). Consider the map

$$\begin{aligned} h : \mathbb{CP}^\infty \times \mathbb{CP}^\infty &\to \mathbb{CP}^\infty \\ ([z_1, z_2, \ldots], [w_1, w_2, \ldots]) &\mapsto [z_1, w_1, z_2, w_2, \ldots]. \end{aligned} \tag{8.1}$$

Consider the line bundle

$$L_{-1} = h^* \mathscr{O}(-1).$$

One can check (exercise) that

$$L_{-1} \cong \pi_1^* \mathscr{O}(-1) \otimes \pi_2^* \mathscr{O}(-1). \tag{8.2}$$

By Künneth, the cohomology ring is given by

$$H^*\left(\mathbb{CP}^\infty \times \mathbb{CP}^\infty\right) \cong \mathbb{Z}[x, y].$$

Let

$$\iota_1 : \mathbb{CP}^\infty \to \mathbb{CP}^\infty \times \{x_0\} \subset \mathbb{CP}^\infty \times \mathbb{CP}^\infty$$

be the inclusion of the first factor. Since $h \circ \iota_1 = \mathbf{1}$, we have

$$\iota_1^* c_1(L_{-1}) = c_1(\mathscr{O}(-1)) = -x.$$

Similarly,

$$\iota_2^* c_1(L_{-1}) = -y.$$

We therefore have

$$c_1\left(L_{-1}\right) = -x - y.$$

Now, given two bundles L and L' over a space X, with classifying maps f and g, respectively, we may take

$$f \times g : X \to \mathbb{CP}^\infty \times \mathbb{CP}^\infty.$$

Then (8.2) gives

$$(f \times g)^*\left(L_{-1}\right) \cong L \otimes L'.$$

So

$$\begin{aligned} c_1(L \otimes L') &= (f \times g)^*(-x - y) \\ &= -\left(f^*(x) + g^*(y)\right) \\ &= c_1(L) + c_1(L'), \end{aligned}$$

as desired.

Note: You can also prove the fact for Chern classes by thinking about the map c_1' defined above using the exponential exact sequence, which involves taking a log...this is what changes multiplication (tensor product) into addition.

Another note: we will later give a different definition of Chern classes for which this fact becomes patently obvious. □

8.3. **First Stiefel-Whitney and Chern class of higher-rank bundles.**

Definition 8.5. If E is a real (resp. complex) vector bundle then the first **Steifel-Whitney class** (resp. **Chern**) class of E is defined by

$$w_1(E) := w_1(\det_{\mathbb{R}} E)$$

or, respectively,

$$c_1(E) = c_1(\det_{\mathbb{C}} E).$$

Since determinant clearly commutes with pullback, these are automatically characteristic classes!

Corollary 8.6. *A real vector bundle is orientable iff* $w_1 = 0$. *A complex vector bundle reduces to* $\mathrm{SU}(n)$ *if and only if* $c_1 = 0$.

Proof. Immediate from Definition 5.1, Proposition 5.4, and Theorem 8.3. □

Proposition 8.7. *For* $r \geq 1$, $w_1(E \oplus F) = w_1(E) + w_1(F)$ *and* $c_1(E \oplus F) = c_1(E) + c_1(F)$.

Proof. If $k = \mathrm{rk} E$ and $\ell = \mathrm{rk}(F)$, then $\bigwedge^{k+\ell}(E \oplus F) = \bigwedge^k E \otimes \bigwedge^\ell F$. We're done by Lemma 8.4. □

One can define the higher Stiefel-Whitney and Chern classes now if one knows the cohomology of the Grassmannian (see *e.g.* Milnor and Stasheff, §6). However, they are not so easy to work with from that perspective. Also, one usually goes the other way, getting the cup product structure on the Grassmannian using the properties of characteristic classes (see Milnor and Stasheff, §7).

Our approach will be to get an explicit construction of the higher Chern classes over smooth manifolds, then to get their properties from the construction. (For Stiefel-Whitney, I'm not sure one can avoid some nasty algebraic topology.)

8.4. **Transversality.** This section develops a key notion in differential topology that is also very handy for dealing with higher-rank vector bundles.

Definition 8.8. Let $f : M \to N$ and $g : Q \to N$ be maps of smooth manifolds. We say f and g are **transverse** and write

$$f \pitchfork g$$

if for all $x \in N$ such that there exist $p \in M$ and $q \in Q$ with $f(p) = x = g(q)$, we have

$$\operatorname{Im} df_p + \operatorname{Im} dg_q = T_x N.$$

On the LHS, we mean the span of two subspaces, not necessarily a direct sum.

Example 8.9. If $f(M) \cap g(Q) = \varnothing$, then they $f \pitchfork g$ by definition. For instance, two skew lines in $\mathbb{R}^3$ are transverse. On the other hand, two lines in $\mathbb{R}^3$ that meet at a point are *not* transverse.

Lemma 8.10. *If $f \pitchfork g$ and both are embeddings, then $f(M) \cap g(Q)$ is an embedded submanifold of N of codimension*

$$\operatorname{codim}(M) + \operatorname{codim}(N).$$

In particular, if

$$\operatorname{codim}(M) + \operatorname{codim}(N) > \dim(Q),$$

i.e.

$$\dim(M) + \dim(N) < \dim(Q),$$

and $f \pitchfork g$, then $f(M) \cap g(Q) = \varnothing$.

Proof. The first statement follows from the implicit function theorem.

For the second statement, assume that $x \in N$ is a transverse intersection point. Then $\operatorname{Im} T_p M + \operatorname{Im} T_q N = T_x N$. But the sum of the dimensions of the spaces on the LHS is less than the dimension of the space on the RHS, so this is absurd. $\square$

Definition 8.11. A section $s : M \to E$ of a vector bundle **vanishes transversely** if $s \pitchfork \underline{0}$, where $\underline{0}$ is the zero section of E.

Theorem 8.12. *Every smooth vector bundle has a section that vanishes transversely.*

Low-brow proof of Theorem 8.12 à la Milnor-Stasheff pages 210–214. Pick non-vanishing sections on trivializing opens, and carefully combine them one-by-one using Sard's Theorem. $\square$

We shall also give a high-brow proof, which requires establishing two lemmas.

Lemma 8.13. *Suppose that $F \to M$ and $\pi : G \to N$ are vector bundles and we have a bundle map $f' : F \to G$ and another space Q with $g' : Q \to G$ and $g : Q \to N$, such that the following diagram commutes:*

$$\begin{tikzcd} F \arrow[r, "f'"] \arrow[d] & G \arrow[d, "\pi"] & \\ M \arrow[r, "f"] & N & Q. \arrow[l, "g"] \arrow[ul, "g'"'] \end{tikzcd}$$

If $f' \pitchfork g'$ then $f \pitchfork g$.

Proof. Diagram chase using fact that π is a submersion (exercise). $\square$

Lemma 8.14. *If $E \to M$ is smooth, then there exists $F \to M$ such that $E \oplus F$ is trivial.*

Proof. If M is compact, then by the argument used in the proof of Theorem 7.3, you can inject E into $\underline{K}^N$, with N finite (since the cover is finite). Then let F be the orthogonal complement of E in $\underline{K}^N$.

For M non-compact, one can use the Whitney embedding theorem to get the statement for a finite N. See Bredon, *Topology and geometry*, p. 97. □

High-brow proof of Theorem 8.12. Let $M \xrightarrow{0} E$ be the zero section. By Lemma 8.14, there is $F \to M$ such that $E \oplus F \cong \underline{K}^N = M \times K^N$. We have a diagram

$$\begin{tikzcd}
F \arrow[r, "f'"] \arrow[d] \arrow[rrr, bend left, "h"] & E \oplus F \arrow[r, "\sim"] \arrow[d, "\pi"] & M \times K^N \arrow[r] & K^N \\
M \arrow[r, "0"] & E & M, \arrow[l, "s"] \arrow[ul, "s'"] \arrow[u, "s''"] &
\end{tikzcd}$$

where f' is a bundle map. Here, we choose s, s', and s'' as follows.

Let h be the composition of the top row. By Sard's Theorem, we may let z be a regular value of h. Let

$$s'' : y \mapsto (y, z).$$

Then let s' and s are defined to make the diagram commute.

We claim that $f' \pitchfork s'$. Identify $E \oplus F \cong M \times K^N$ by the middle map in the top row, which gives

$$T_y(E \oplus F) \cong T_x M \oplus K^N,$$

for each $y \in E \oplus F$ with $\pi(y) = x$. Suppose $v \in F_x$ with $f'(v) = (x, z) \in M \times K^N$. Then

$$T_{f'(v)}(E \oplus F) \cong T_x M \oplus T_z K^N.$$

The image of df'_v spans the second factor $T_z K^N \cong K^N$ because z is chosen to be a regular regular value of h. We have $ds'' = (\mathbf{1}, 0)$ by construction, so ds'' spans $T_x M$. Therefore, the two together span $T_{f'(v)}(E \oplus F)$. We conclude that $f' \pitchfork s'$.

By Lemma 8.13, we can conclude that $\underline{0} \pitchfork s$, so s vanishes transversely. □

Remark 8.15. A slight generalization of the previous argument shows that for *any* map f from a manifold X to the total space E (in place of the zero section), there exists a section of E transverse to f. We can then get the following statement: given any immersed submanifold $X \subset N$ and map $f : M \to N$, there exists an immersed submanifold X' close to X that is transverse to f. This follows by letting E be the normal bundle of X and using the tubular neighborhood theorem.

Among other things, this result is the basis for "intersection theory:" see Griffiths and Harris, §0.4.

8.5. **Vector bundles on manifolds of dimension ≤ 4.**

Theorem 8.16. *Every complex vector bundle E of rank r on M^2 or M^3 splits as*

$$L \oplus \underline{\mathbb{C}}^{r-1}$$

for a complex line bundle L.

Proof. Let $d = \dim M = 2$ or 3.

If $r = 1$, there is nothing to show.

Suppose $r \geq 2$. By the last theorem, we may choose a transversely vanishing section s of E. The image of the zero section and of s both have dimension d, whereas $\dim E = d + 2r$. But

$$d + d < d + 2r,$$

since $r \geq 2$ and $d \leq 3$. By Lemma 8.10, we conclude that $s \cap \underline{0} = \varnothing$. Hence, s spans a trivial subbundle of E. This gives $E \cong E' \oplus \mathbb{C}$.

If $\operatorname{rank} E' \geq 2$, we may choose a nonvanishing section of E', giving another trivial factor. Continuing, we obtain the desired splitting. ☐

Corollary 8.17. *The first Chern class classifies complex vector bundles of any rank on M^2 or M^3.*

Proof. By the previous Theorem, we in fact have

$$\det E \cong L$$

for the line bundle L appearing in the splitting. So the result follows from the corresponding result for line bundles. ☐

Theorem 8.18. *If M is a compact oriented 4-manifold, then the first Pontrjagin class gives an isomorphism*

$$\tilde{p}_1 : \{\mathrm{SU}(2) \text{ bundles on } M\} \to H^4(M) \cong \mathbb{Z}.$$

Proof. Recall that SU(2)-bundles are the same as quaterionic line bundles, so they are classified by $\mathbb{HP}^\infty$. Let $f : M \to \mathbb{HP}^\infty$ be the classifying map of such a bundle.

The cell structure of $\mathbb{HP}^\infty$ has a single cell in each dimension divisible by 4; consequently, the subcomplex $\mathbb{HP}^1 \cong S^4$. By cellular approximation, $f \sim g$ for some $g : M \to \mathbb{HP}^1 = S^4$. By a theorem of Hopf, for any compact oriented n-manifold N, there is an isomorphism $[N, S^n] \xrightarrow{\cong} \mathbb{Z}$ by $[f] \mapsto \deg(f)$, so $\tilde{p}_1$ is an isomorphism. ☐

Remark 8.19. For a proof of the Hopf Theorem and its generalization to $\dim(M) > n = \dim S^n$ via framed cobordism theory (based mainly on the transversality theorem discussed in the previous subsection), see Milnor's classic short text *Topology from the differentiable viewpoint.*

Corollary 8.20. *$SU(n)$-bundles over a 4-manifold M all split as*

$$E = E' \oplus \underline{\mathbb{C}}^{r-2},$$

where E' is an SU(2)*-bundle. In particular, if the M is orientable,* SU(n)*-bundles are classified by a single integer.*

Proof. Exercise. ☐

Remark 8.21. By Theorem 8.18,

$$\{\mathrm{SU}(2)\text{-bundles on } S^7\} \cong \pi_6(S^3) \cong \mathbb{Z}_{12}.$$

However, the first Pontriyagin class of a bundle on S^7 is zero, simply because $H^4(S^7) = 0$; hence, $\tilde{p}_1$ doesn't classify SU(2)-bundles on S^7. This shows that characteristic classes, while a handy device in low dimensions/ranks, in general do not classify bundles up to isomorphism.

8.6. **Exercises.**

1. Check that $L_{-1} \cong \pi_1^*\mathscr{O}(-1) \otimes \pi_2^*\mathscr{O}(-1)$, where $L_{-1} = h^*\mathscr{O}(-1)$ for the map defined by (8.1).
2. For a complex vector bundle E, prove that $c_1(E^*) = -c_1(E)$.
3. Prove Lemma 8.13.
4. Prove Corollary 8.20.

9. Principal bundles, reduction of structure group (2/22)

9.1. **Definition and examples.** Principal bundles give a way to make the structure group intrinsic, rather than something tied to a collection of local frames.

Definition 9.1. Let G be a Lie group. A **principal bundle** with structure group G, also called a (principal) G-bundle, is a space P with a continuous, free, right G-action such that the projection map

$$\pi : P \to X = P/G$$

is locally trivial; that is, for all $x \in X$ there is $U \ni x$ open and a G-equivariant homemorphism φ such that

$$\begin{tikzcd} U \times G \arrow[r, "\varphi"] \arrow[dr] & \pi^{-1}(U) \arrow[d] \\ & U \end{tikzcd}$$

commutes.

Examples 9.2.

- Trivial bundle $X \times G$ with the action of G on itself by right-multiplication.
- Let $P = S^1$, $X = S^1$ and $\pi : P \to X$, $z \mapsto z^2$. This is a principal $\mathbb{Z}_2$-bundle. The action of $\mathbb{Z}_2$ on S^1 is negation, which acts transitively on each fiber $\{\pm z\}$. This is the boundary of the Möbius strip.
- If G is a Lie subgroup of a group K, then $\mathrm{K} \to \mathrm{K}/G$ (right coset space) is a principal G-bundle.
- (Hopf fibration) Recall that $\mathrm{U}(1) \cong S^1$, $\mathrm{SU}(2) \cong S^3$ and that S^1 acts on $S^3 \subset \mathbb{C}^2$ by multiplication. Ths quotient is $\mathrm{SU}(2)/\mathrm{U}(1) = \mathbb{CP}^1 \cong S^2$. So we obtain a principal bundle $P = S^3 \to \mathbb{CP}^1$ with structure group U(1), which identical with the *Hopf fibration*

$$S^1 \to S^3 \to S^2.$$

Alternatively, this can be viewed as the unit circle bundle in $\mathbb{C} \to \mathscr{O}(-1) \to \mathbb{CP}^1$, since

$$\mathscr{O}(-1) \setminus \underline{0} \cong \mathbb{C}^2 \setminus \{(0,0)\}.$$

- (Unit circle bundle inside any complex line bundle) To generalize the previous example, let L be a complex line bundle (with a hermitian metric), and let $P \subset L$ be the unit circle bundle. The action of unit complex scalars $U(1)$ on L preserves P, and acts freely and transitively on each fiber

$$P_x \cong S^1 \subset L_x \cong \mathbb{C}.$$

It is obvious that the local trivializations of L give rise to local trivializations of P, so this is a principal U(1)-bundle.

- (Generalized Hopf fibration) In particular, for $L = \mathscr{O}(-1) \to \mathbb{CP}^n$, we have

$$\begin{tikzcd}
\mathrm{U}(1) \cong S^1 \arrow[r] \arrow[d, hook] & P = S^{2n+1} \arrow[r, hook] \arrow[d, hook] \arrow[dr] & \mathbb{C}^{n+1} \setminus \{(0,0)\} \arrow[d] \\
\mathbb{C} \arrow[r] & \mathscr{O}(-1) \arrow[r] & \mathbb{CP}^n
\end{tikzcd}$$

so we obtain a principal bundle with structure group $U(1)$, whose total space is actually the sphere S^{2n+1}.

- (Unit SU(2)-bundle inside a quaternionic line bundle) By similarly considering the unit sphere bundle inside a quaternionic line bundle, with the global right-action of unit quaternionic scalars $(\mathrm{SU}(2) \subset \mathbb{H}^*)$, we obtain a principal SU(2)-bundle.
- (Quaternionic Hopf fibration) For example, for the tautological bundle over $\mathbb{HP}^n$, we have

$$\begin{tikzcd}
S^3 \arrow[r] \arrow[d, hook] & S^{4n+3} \arrow[d, hook] \arrow[dr] & \\
\mathbb{H} \arrow[r] & \mathscr{O}_{\mathbb{HP}^n}(-1) \arrow[r] & \mathbb{HP}^n
\end{tikzcd}$$

The first of these is sometimes called the *quaternionic Hopf fibration*

$$S^3 \to S^7 \to S^4.$$

This actually gives a generator of $\mathbb{Z} \subset \pi_7(S^4)$.

- (Covering spaces) For an example of a completely different flavor, let $p : \tilde{X} \to X$ be any covering map of a path-connected space. Let

$$G = \{\text{Deck transformations}\} = \pi_1(X)/p_*\pi_1(\tilde{X})$$

with the discrete topology. Then $p : \tilde{X} \to X$ is a principal G-bundle. Notice that the identification of the fibers with G is non-canonical: if we change the basepoint, then it changes. So this demonstrates the idea that the fiber of a principal bundle is homeomorphic to G, due to having a free, transitive G-action, but not in a canonical way. (This is called a "G-torsor" in fancy language.)

Definition 9.3. A principal bundle $\pi : P \to X$ is called **smooth** if π is a submersion and the G-action is smooth.

Theorem 9.4. *Every free, proper, smooth G-action on a smooth manifold P gives a smooth principal bundle $P \to P/G$.*

Proof sketch. The Quotient Manifold Theorem (Lee, *Smooth manifolds*, Chapter 21) implies that π is a submersion. Hence, we have a local section $s : U \to P$ by the implicit function theorem, yielding a trivialization

$$\begin{aligned} U \times G &\to P|_U \\ (x,g) &\mapsto s(x)\cdot g. \end{aligned}$$

□

Note: if G is compact, then any continuous action is proper. The previous theorem says that any free action by a compact Lie group on a smooth manifold gives a principal bundle. So we are dealing with a very general concept!

9.2. **Transition functions, bundle maps, and isomorphisms.**

Lemma 9.5. *A right G-equivariant map $\varphi : G \to G$ is given by left-multiplication by some $g_0 \in G$.*

Proof. If $g_0 = \varphi(e)$, then $\varphi(g) = \varphi(eg) = \varphi(e)g = g_0 g$. □

Proposition 9.6. *The transition functions of P are left-multiplication by G-valued functions $g_{ab}(x)$ satisfying $g_{bc}g_{ab} = g_{ac}$ and $g_{aa} = e$.*

Proof. The transition map

$$\varphi_b^{-1} \circ \varphi_a : U_a \cap U_b \times G \to U_a \cap U_b \times G.$$

is G-equivariant on each fiber over $U_a \cap U_b$, so by Lemma 9.5, it is of the form $(x,g) \mapsto (x, g_{ab}(x)\cdot g)$. The remainder of the statement is immediate from the cocycle conditions. □

Definition 9.7. A **bundle map** $\varphi : P \to Q$ is a right G-equivariant map.

Note that φ automatically preserves fibers, and by Lemma 9.5, is given by left-multiplication in the trivializations.

Proposition 9.8. *A principal bundle P is trivial over $U \subset X$ if and only if there exists a section $s : U \to P$.*

Proof. Given a section $s : U \to P|_U$, we obtain a homeomorphism $U \times G \to P|_U$ by $(x,g) \mapsto s(x)\cdot g$. □

Definition 9.9. A **bundle isomorphism** is a bundle map over X; that is, a bundle map $P \to Q$ such that

$$\begin{tikzcd} P \arrow[r] \arrow[dr] & Q \arrow[d] \\ & X \end{tikzcd}$$

commutes.

9.3. **Bundle operations.** Some of our old bundle operations still work:

- Pullback: the same, and the correspondence between pullback and bundle maps still holds
- Product: if $P \to X$ is a G-bundle and $Q \to X$ is an H-bundle, then $P \times_X Q$ is a $G \times H$-bundle over X, analogous to direct sum of vector bundles.

We also have a new kind of product for principal bundles, which is what makes them so useful.

Definition 9.10. Let F be a topological space, and suppose G acts on F on the left; that is, there is a map

$$\rho : G \to \mathrm{Homeo}(F)$$

such that the evaluation map

$$\begin{aligned} G \times F &\to F \\ (x, y) &\mapsto \rho(x) \cdot y \end{aligned}$$

is continuous. The **associated (fiber) bundle** (with fiber F) is

$$\begin{aligned} P \times_\rho F &= (P \times F)/\sim, \\ (p, f) &\sim (p \cdot g, \rho(g^{-1}) \cdot f). \end{aligned}$$

Note that this is no longer necessarily a principal bundle, just a *fiber bundle*, *i.e.* a (locally trivial) fibration with fiber F.

Often, when the action $\rho : G \to \mathrm{Aut}(F)$ is obvious (such as $\rho : \mathrm{U}(1) \to \mathbb{C}^* \circlearrowright \mathbb{C} = F$), we will simply write

$$P \times_G F$$

instead of $P \times_\rho F$, which is standard notation.

Lemma 9.11. *Suppose that $P = U \times G$ is a trivial principal bundle. Then the map*

$$\begin{aligned} (U \times G) \times_\rho F &\xrightarrow{\sim} U \times F \\ [(x, g), f] &\mapsto (x, \rho(g) \cdot f) \end{aligned}$$

is an isomorphism.

Proof. It's well-defined because

$$[(x, g \cdot h), \rho(h^{-1})f] \mapsto (x, \rho(gh)\rho(h^{-1}) \cdot f) = (x, \rho(g) \cdot f),$$

and the inverse is $(x, f) \mapsto [(x, e), f]$. □

Immediately from the lemma, we have:

Corollary 9.12. *$P \times_\rho F$ is a fiber bundle with fiber F and transition functions $\rho(g_{ab})$.*

Example 9.13. Suppose $F = V$ is a finite-dimensional vector space and $\rho : G \to \mathrm{GL}_K(V)$ is a K-linear representation. Then $P \times_\rho V$ is a vector bundle over K of rank $r = \dim_K V$.

For instance, if P is the principal U(1)-bundle given as the unit circle bundle inside a line bundle, L (per Example 9.2), then we have a natural isomorphism $P \times_{\mathrm{U}(1)} \mathbb{C} \cong L$. (Exercise)

Definition 9.14. The **bundle of gauge transformations** of P is

$$\mathscr{G}_P = P \times_\rho G,$$

where $\rho(g) \cdot h = ghg^{-1}$ (the adjoint action).

The gauge group is a fiber bundle with fiber G. However, it is *not* naturally a principal G-bundle. One good reason why not is that $\mathscr{G}_P$ always has a section, so that would make it always trivial. Note however that for abelian structure groups, $\mathscr{G}_P$ is always globally trivial (since the adjoint action is trivial).

Proposition 9.15. *We have*

$$\Gamma(U, \mathscr{G}_P) = \{\textit{bundle isomophisms of } P \textit{ over } U\}.$$

Proof. Exercise. □

Definition 9.16. Let $\mathfrak{g} = \mathrm{Lie}(G)$ and $\rho : G \to \mathrm{Aut}(\mathfrak{g})$ the adjoint rep. Then

$$\mathfrak{g}_P = P \times_\rho \mathfrak{g}$$

is the bundle of **infinitesimal gauge transformations** of P.

The infinitesimal gauge transformations form a vector bundle (*cf.* Example 9.13). The exponential map $\mathfrak{g} \to G$ is equivariant (*i.e.* an intertwiner) for the adjoint actions of G. It therefore gives rise to a global exponential map

$$\exp : \mathfrak{g}_P \to \mathscr{G}_P$$

which is a homeomorphism between a neighborhood of the zero section and a neighborhood of the identity section. The map exp also works on spaces of sections.

Remark 9.17 (*Vector bundles are enough*)**.** It is a fact that any compact Lie group has a faithful representation $\rho : G \hookrightarrow \mathrm{Aut}_{\mathbb{C}}(V)$. Since ρ is injective, the transition functions of the vector bundle $P \times_\rho V$ encode those of P. Hence, if you don't like principal bundles and only care about compact structure groups, you are free to just study complex vector bundles (with structure group G, in the sense of §5.1) without any loss of generality.

9.4. **Extension and reduction of structure group.**

Definition 9.18. Let $H < G$ be a subgroup.

- If Q is an H-bundle, then $Q_G = Q \times_H G$ is the **extension** of Q to G.
- If P is a G-bundle, then an H-bundle Q is a **reduction** of P to H if there is an isomorphism $Q_G \to P$.

Notice that every principal G-bundle has in particular an action by $H < G$. We make the following observation:

Lemma 9.19. *G-isomorphisms $Q_G \to P$ are equivalent to H-equivariant maps $Q \to P$ over X (i.e. sending Q_x into P_x for all $x \in X$).*

Proof. ($\to$) Take the composition $Q \hookrightarrow Q_{\mathrm{G}} \to P$.

($\leftarrow$) Suppose $P \to X$ is a G-bundle and $Q \to X$ an H-bundle and that $\varphi : Q \to P$ is an H-equivariant map over X. Then φ factors as

$$\begin{tikzcd} Q \arrow[r, "\varphi_1"] \arrow[rr, bend left, "\varphi"] \arrow[dr] & Q_G \arrow[r, "\varphi_2"] \arrow[d] & P \arrow[dl] \\ & X & \end{tikzcd}$$

where

$$\begin{aligned} \varphi_1 : Q \to Q_G, &\quad x \mapsto [x, e] \\ \varphi_2 : Q_G \to P, &\quad [x, g] \mapsto \varphi(x) \cdot g. \end{aligned}$$

The latter map is well-defined because

$$[xh, h^{-1}g] \mapsto \varphi(xh) \cdot h^{-1}g = \varphi(x) \cdot hh^{-1}g = \varphi(x)g$$

for $h \in H$. $\square$

Proposition 9.20. *Let P be a G-bundle. Reductions of P to H are equivalent to (in particular, in bijection with) sections of $P \times_{\mathrm{G}} G/H$.*

Proof. First observe that we have an isomorphism

$$\begin{aligned} P \times_{\mathrm{G}} G/H &\to P/H \\ [p, g] &\mapsto (p \cdot g) \cdot H. \end{aligned} \tag{9.1}$$

The result comes from the diagram:

$$\begin{tikzcd} Q \arrow[r] \arrow[dr] & P \arrow[r] \arrow[d] & P/H \cong P \times_G G/H \arrow[dl] \\ & X \arrow[ur, bend right, "s"'] & \end{tikzcd}$$

Given a reduction of structure group Q, s comes from the composition of the top row, which descends to X (because the orbits of H on Q are clearly sent to points in P/H).

Given a section $s : X \to P \times_{\mathrm{G}} G/H$, we can set $Q = s^* P$, where P is regarded as an H-bundle over P/H. $\square$

9.5. **Exercises.**

1. Prove Proposition 9.15.
2. Let P be the (trivial) $U(1)$-bundle over S^1. Give an example of two reductions to $\mathbb{Z}_2 = \{\pm 1\} \subset \mathrm{U}(1)$ that are not isomorphic as $\mathbb{Z}_2$-bundles.
3. Check that the transition functions of an associated bundle are given by $\{\rho(g_{ab})\}$, where $\{g_{ab}\}$ are the transition functions of P, as stated in Corollary 9.12.
4. (From Example 9.13) Let L be a complex line bundle with a Hermitian metric, and P the unit sphere bundle viewed as a principal $\mathrm{U}(1)$-bundle. Write down an isomorphism $P \times_{\mathrm{U}(1)} \mathbb{C} \cong L$.
5. Use the equivalence statement in Proposition 9.20 to give proofs of Propositions 5.2 and 5.4.

10. Correspondence with vector bundles, universal G-bundles (2/24)

10.1. **Correspondence between principal G-bundles and vector bundles.** We previously defined a vector bundle with structure group G in terms of transition functions. Principal G-bundles give us a nicer way to rephrase this.

Example 10.1. If L is a complex line bundle with a Hermitian metric, then the unit sphere bundle is a principal U(1)-bundle P. On the other hand, $P \times_{U(1)} \mathbb{C} \cong L$ (here, we write $P \times_{U(1)} \mathbb{C}$ to denote $P \times_\rho \mathbb{C}$ where $\rho : U(1) \to \mathbb{C}^*$ is the standard representation).

Definition 10.2. Given $E \to X$ a real (resp. complex) vector bundle with a (Hermitian) metric, the **principal frame bundle** $\mathrm{Fr}(E)$ is the principal $O(r)$-bundle (resp. $U(r)$-bundle) with fiber

$$\mathrm{Fr}(E)_x = \{(v_1, \dots, v_r) \in E_x \text{ orthonormal}\} \subset E_x^{\oplus r}$$

over $x \in X$. An element $a^\beta{}_\alpha \in O(r)$ (resp. $U(r)$) acts on the right by

$$a^\beta{}_\alpha : (v_\alpha) \mapsto (v_\beta a^\beta{}_\alpha).$$

Proposition 10.3. *Let $E \to X$ be a K-vector bundle of rank r.*

- *If $K = \mathbb{R}$, then* $\mathrm{Fr}(E) \times_{\mathrm{O}(r)} \mathbb{R}^r \cong E$.
- *If $K = \mathbb{C}$, then* $\mathrm{Fr}(E) \times_{\mathrm{U}(r)} \mathbb{C}^r \cong E$.

Proof. The isomorphism is

$$[(v_\alpha), (a^\alpha)] \mapsto \sum_\alpha v_\alpha a^\alpha.$$

This is almost manifestly well-defined, and an isomorphism. □

Suppose that E is a vector bundle with structure group G. Concretely, this means that all fibers of E are isomorphic to a vector space V and there is a map $\rho : G \to \mathrm{GL}(V)$ such that the transition maps lie in the image. Since $\rho(G)$ is a subgroup of $\mathrm{GL}_K(V) \subset V \otimes V^*$, it is stable under the right action $\rho(h) \cdot g := \rho(h)\rho(g)$.

Definition 10.4. Let $E \to X$ be a vector bundle with structure group G, V a fiber of E, and $\rho : G \to \mathrm{GL}_K(V)$ as above. The **frame bundle** of E is the principal G-bundle $\mathrm{Fr}_G(E) \subset E \otimes \underline{V}^*$ with fiber

$$\mathrm{Fr}_G(E)_x = \rho(G) \subset E_x \otimes V^* \cong V \otimes V^*$$

and G-action by right-multiplication (*i.e.*, acting trivially on E_x and on V^* by the dual representation).

Proposition 10.5. $\mathrm{Fr}_G(E) \times_G V \cong E$

Proof. Exercise in the definitions. □

10.2. **Homotopy classification of principal bundles.** Previously, we classified vector bundles using maps to Grassmannians. We'll now do the same for principal bundles. The classifying space for principal G-bundles is called BG, and the universal principal G-bundle is called $EG \to BG$.

Theorem 10.6 (Homotopy theorem for principal bundles)**.** *If $P \to X \times [0,1]$ is a principal G-bundle on paracompact base X, then $P \cong \pi_1^* P$, where $\pi_1 : X \times [0,1] \to X \times \{1\}$.*

Proof. Choose a countable cover $\{U_a\}$ of X such that $P|_{U_a \times [0,1]}$ is trivial for each a (exercise, sketched in class). Let ρ_a be a locally finite partition of unity for $\{U_a\}$. Let

$$\lambda_a(x) = \frac{\rho_a(x)}{\max_b(\rho_b(x))}.$$

For each $x \in X$, we evidently have $\max_a \lambda_a(x) = 1$.

Let

$$r_a : X \times [0,1] \to X \times [0,1], \quad (x,t) \mapsto (x, \max(t, \lambda_a(x))).$$

Since

$$r_a|_{X \smallsetminus U_a \times [0,1]} = \mathbf{1},$$

r_a lifts to a bundle map $\tilde{r}_a : P \to P$ defined by

$$\tilde{r}_a|_{X \smallsetminus U_a \times [0,1]} = \mathbf{1}$$

on U_a^c, and on U_a as the composition

$$P|_{U_a \times [0,1]} \cong U_a \times [0,1] \times G \overset{(r_a, \mathbf{1})}{\to} U_a \times [0,1] \times G \cong P|_{U_a \times [0,1]}.$$

Now, consider the infinite composition

$$r = \cdots \circ r_3 \circ r_2 \circ r_1$$

(sensical because the partition of unity is locally finite, so only finitely many r_i's act by the identity on any point). This is covered by

$$\tilde{r} = \cdots \circ \tilde{r}_3 \circ \tilde{r}_2 \circ \tilde{r}_1.$$

But since $\max_a \lambda_a(x) = 1$ for any x, r factors as

$$\begin{tikzcd} & X \times \{1\} \arrow[d, hook] \\ X \times [0,1] \arrow[ur] \arrow[r, "r"] & X \times [0,1] \end{tikzcd}$$

so $\tilde{r}$ is the desired bundle map covering the projection $X \times [0,1] \to X \times \{1\}$. □

Corollary 10.7. *If $f_0 \sim f_1$, then $f_0^* P \cong f_1^* P$.*

Definition 10.8. $EG \to BG$ is called **universal** for G-bundles if for every G-bundle $P \to X$, there exists a bundle map $P \to EG$, which is unique up to homotopy *of G-equivariant maps* (*i.e.* bundle maps).[17]

By the same argument as for universal vector bundles, BG is automatically unique up to homotopy equivalence.

Theorem 10.9. *A G-bundle $Q \to Y$ is universal iff the total space Q is contractible (as a space without G-action).*

[17] My definition of the universal G-bundle during class was slightly too weak. In fact, my argument for the ($\to$) direction of Theorem 10.9 was incorrect without this stronger definition.

Proof sketch. See tom Dieck's *Algebraic topology*, Theorem 14.4.12, for a short, complete, and immoral, proof. See Steenrod, Theorem 19.4, for a proof of a bit weaker statement.

($\rightarrow$) To see that Q is weakly contractible; that is, $\pi_i(Q) = 0$, let $\varphi : S^n \to Q$. We obtain a bundle map $\tilde{\varphi} : S^n \times G \to Q$ by $(x, g) \mapsto \varphi(x) \cdot g$. Meanwhile, we can also make a bundle map by fixing $q_0 \in Q$ and letting $\varphi_0 : S^n \times G \to Q$ by $(x, g) \mapsto q_0 \cdot g$. Since Q is universal, $\tilde{\varphi} \sim \varphi_0$ through bundle maps $\varphi_t : (S^n \times G) \times [0, 1] \to Q$ with $\varphi_1 = \tilde{\varphi}$ and $\varphi_0 = \varphi_0$.

Now, the restriction of φ_t to $S^n \times \{e\} \times [0, 1] \to Q$ gives a homotopy contracting φ.

($\leftarrow$) Suppose that Q is contractible. Given a CW complex X and a bundle $P \to X$, we obtain a map $P \to Q$ using the fact that P is trivial on every cell, as follows. Suppose that the map has been constructed up to the n-skeleton of X. Let B^{n+1} be an n-cell with boundary S^n. We obtain the diagonal dashed arrow below because Q is contractible, which we can then lift to $P|_{B^{n+1}} \cong B^{n+1} \times G$ by multiplication, as usual.

$$\begin{tikzcd}
P|_{S^n} \arrow[r, hook] \arrow[d] \arrow[rr, bend left] & P|_{B^{n+1}} \arrow[r, dashed] \arrow[d] & Q \arrow[d] \\
S^n \arrow[r, hook] & B^{n+1} \arrow[r] \arrow[ur, dashed] & X.
\end{tikzcd}$$

This shows how to lift the bundle map to the $n+1$-skeleton, and it continues in the same way. □

Corollary 10.10. *BG is connected, and we have $\pi_{i+1}(BG) = \pi_i(G)$, for all $i \geq 0$.*

Proof. This follows from the long exact homotopy sequence of the fibration

$$G \to EG \to BG$$

and the fact that $\pi_i(EG) = 0$ for all i. □

Corollary 10.11. *Given a subgroup $H < G$, the total space of EG is also the total space of EH.*

Proof. We have

$$\begin{tikzcd}
EG \arrow[d] \arrow[dr] & \\
BH = EG/H \arrow[r] & BG = EG/G.
\end{tikzcd}$$

Since EG is contractible, the principal bundle at left must be $EH \to BH$! □

Example 10.12. Consider $\mathbb{Z}_2 < \mathrm{U}(1) < \mathrm{SU}(2)$. We have

$$\begin{tikzcd}
 & S^\infty = E\mathbb{Z}_2 = E\mathrm{U}(1) = E\mathrm{SU}(2) \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathbb{RP}^\infty = B\mathbb{Z}_2 \arrow[r] & \mathbb{CP}^\infty = B\mathrm{U}(1) \arrow[r] & \mathbb{HP}^\infty = B\mathrm{SU}(2)
\end{tikzcd}$$

Here, because S^∞ is contractible, we know that it is the universal bundle for each of these groups, so we get the identities in the bottom row. (These also follow simply because we have demonstrated a canonical equivalence between each of the three types of line bundle and the corresponding type of principal bundle.)

Theorem 10.13 (Milnor)**.** *For any topological group G, $EG \to BG$ exists. And if G is a Lie group, then BG has the homotopy type of a CW complex.*

Proof idea. In the previous example, we can write $S^\infty \subset \mathbb{C}^\infty$ as

$$S^\infty = \{(z_1 t_1, z_2 t_2, \dots,) : t_i \in [0,1], z_i \in \mathrm{U}(1), \sum t_i^2 = 1\},$$

where all but finitely many t_i's must be zero, and for $t_i = 0$, all z_i's are considered the same. Generalizing this, Milnor builds a space called the *infinite join* of G with itself:

$$\begin{aligned} EG &= G * G * G * \cdots \\ &= \left\{ ((g_1, t_1), (g_2, t_2), \dots) : g_i \in G, t_i \in [0,1], \sum t_i^2 = 1 \right\}. \end{aligned}$$

One can let G act here on the right, which is clearly a free action. See tom Dieck, §14.4, for the proof that this guy is universal (and contractible, which he uses to prove that actually any EG must be contractible). □

Example 10.14.

$$\begin{tikzcd} E\mathrm{SO}(n) \arrow[r] \arrow[d, "="] & B\mathrm{SO}(n) = \widetilde{G_n(\mathbb{R}^\infty)} \arrow[d, "/\mathbb{Z}_2"] \\ E\mathrm{O}(n) \arrow[r] & B\mathrm{O}(n) = G_n(\mathbb{R}^\infty). \end{tikzcd}$$

Here, $B\mathrm{SO}(n)$ has to be there by the previous argument. In fact, it is the connected double cover of $B\mathrm{O}(n)$, which must exist because $\pi_1(B\mathrm{O}(n)) = \pi_0(\mathrm{O}(n)) = \mathbb{Z}_2$. More explicitly, this is just the Grassmannian of *oriented* n-planes, which you could show classifies $\mathrm{SO}(n)$-bundles by carrying out the same proof as for $\mathrm{O}(n)$-bundles while keeping track of orientation. So in the end, I don't have any neat consequences of Milnor's theorem to show you. (Except maybe that EG is always strongly contractible, which perhaps you really can't prove without constructing it.)

10.3. **Exercises.**

1. Think through the proof of Proposition 10.5.
2. Justify the first sentence in the proof of Theorem 10.6 (as sketched during class). Note: the fact that you can choose a *countable* collection of trivializations is proven in the Appendix to Ch. 1 of Hatcher's VBKT.
3. * Find a neat consequence of Milnor's Theorem 10.13, the existence of classifying spaces.

Part II. Connections, curvature, Chern-Weil Theory

11. Connections (3/1)

11.1. **The definition.** A **connection**, A, on a smooth vector bundle $E \to M$ assigns to each piecewise smooth path $\gamma : [a,b] \to M$ and $w \in E_{\gamma(a)}$ a piecewise smooth lift

$$\tilde{\gamma}_w^A = \tilde{\gamma}_w : [a,b] \to E$$

such that

$$\tilde{\gamma}_w(a) = w$$

and

$$\pi_E \circ \tilde{\gamma}_w = \gamma.$$

This assignment must satisfy:

(1) If $\gamma : [a,b] \to M$ and $\eta : [b,c] \to M$, with $\gamma(b) = \eta(b)$, then

$$\widetilde{\gamma \star \eta} = \tilde{\gamma} \star \tilde{\eta}.$$

(2) The map

$$\begin{aligned} E_{\gamma(a)} &\to E_{\gamma(b)} \\ w &\mapsto \tilde{\gamma}_w(b) \end{aligned}$$

is K-linear.

(3) The derivative $\tilde{\gamma}_w'(a) \in T_w E$ is a linear function of $\gamma'(a) \in T_{\pi(w)} M$.[18]

Remark 11.1. If one drops (2), then this definition makes sense on a general smooth fiber bundle (this is called an "Ehresmann connection"). In fact, one just has to choose a "horizontal distribution" consisting of the subbundle of TE given at $w \in E$ by

$$\{\tilde{\gamma}_w'(0) \mid \gamma(0) = \pi_E(w)\}.$$

11.2. **Terminology.**

- The linear map

$$\begin{aligned} T^A_{\gamma,a,b} : E_{\gamma(a)} &\to E_{\gamma(b)} \\ w &\mapsto \tilde{\gamma}_w(b) \end{aligned}$$

is called the **parallel transport** operator along γ.

- Let

$$\Omega_x = \{\gamma : [0,1] \to M \mid \gamma(0) = \gamma(1) = x\}$$

be the space of loops based at x. The group

$$\{T^A_{\gamma,0,1} \mid \gamma \in \Omega_x\} \subset \mathrm{GL}_K(E_x)$$

is called the **holonomy group** of A at x. It can be shown using (3) above that this is indeed a group (exercise).

[18] This is really two statements: that $\tilde{\gamma}_w'(a)$ depends only on the initial tangent vector to the path γ, and that it does so in a linear fashion.

- The **restricted holonomy group** at x is given by

$$\{T_{\gamma,0,1} \mid \gamma \in \Omega_x \text{ is contractible}\}.$$

- A connection is called **flat** if the restricted holonomy groups are trivial.
- Given a section s of E along γ, the **covariant derivative** of s *along* γ is defined by

$$\frac{\nabla_A s(\gamma(t))}{dt} := \lim_{h\to 0}\left(\frac{s(\gamma(t+h)) - T^A_{\gamma,t,t+h}\big(s(\gamma(t))\big)}{h}\right) \in E_{\gamma(t)}.$$

  Note that this is again a section of E along γ.

Proposition 11.2. *The covariant derivative along γ satisfies the Leibniz rule:*

$$\frac{\nabla_A(\lambda s)}{dt} = \frac{d\lambda}{dt}\cdot s + \lambda \cdot \frac{\nabla_A s}{dt}$$

for $\lambda \in C^\infty$ and $s \in \Gamma(E)$.

Proof. By (1), it suffices to check for $t = 0$:

$$\begin{aligned}
\frac{\nabla_A(\lambda\cdot s)}{dt}\Big|_{t=0} &= \lim_{t\to 0}\frac{1}{t}\Big(\lambda(\gamma(t))\cdot s(\gamma(t)) - T^A_{\gamma,0,t}\big(\lambda(\gamma(0))\cdot s(\gamma(0))\big)\Big)\\
&= \lim_{t\to 0}\frac{1}{t}\Bigg(\Big[\lambda(\gamma(t))\cdot s(\gamma(t)) - \lambda(\gamma(0))\cdot s(\gamma(t))\Big] + \lambda(\gamma(0))\cdot\Big[s(\gamma(t)) - T^A_{\gamma,0,t}(s(\gamma(0)))\Big]\Bigg)\\
&= \frac{d\lambda}{dt}\Big|_{t=0} s(\gamma(0)) + \lambda(\gamma(0))\frac{\nabla_A s}{dt}\Big|_{t=0}.
\end{aligned}$$

□

Definition 11.3. A **covariant derivative** (without any paths!) is a map

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$$

which is linear with respect to multiplication by constants, and satisfies the Leibniz rule

$$\nabla(\lambda\cdot s) = d\lambda \otimes s + \lambda\cdot\nabla s,$$

for scalar functions $\lambda \in C^\infty(M)$ and sections $s \in \Gamma(E)$.
If v is a tangent vector at $x \in M$, we can pair with $\nabla(s) \in T^*M\otimes E$ to obtain an element

$$\nabla_v(s) := (\nabla s)(v) \in E_x.$$

This is thought of as the "covariant derivative in the v direction." Similarly, for a vector field X and $s \in \Gamma(E)$, we obtain another section of E denoted by

$$\nabla_X s := (\nabla s)(X).$$

Note that just by definition, we have

$$\nabla_{\lambda X} s = \lambda \nabla_X s \tag{11.1}$$

for any $\lambda \in C^\infty$.

11.3. **Connections and covariant derivatives are the same thing.** Given a connection, A, we can define a covariant derivative

$$\nabla_A : \Gamma(E) \to \mathrm{Hom}(TM, E) \cong T^*M \otimes E$$
$$s \mapsto \left(v \in T_x M \mapsto \frac{\nabla_A s(\gamma(t))}{dt}\Big|_{t=0} \in E_x \right)$$

where $\gamma : [0,1] \to M$ is any path with $\gamma'(0) = v$. By (3), this is independent of the path, and we already showed that it satisfies Leibniz rule.

Conversely, given any covariant derivative ∇, we obtain a connection by defining the lift $\tilde{\gamma}_w(t)$ to satisfy

$$\nabla_{\gamma'(t)} \tilde{\gamma}_w(t) = 0, \quad \tilde{\gamma}_w(0) = w. \tag{11.2}$$

We can sum up the two-way correspondence just by saying that a section $s(\gamma(t))$ is covariantly constant along $\gamma(t)$ if and only if

$$s(\gamma(t)) = \tilde{\gamma}_{s(0)}(t).$$

In the subsection that follows, we will use local coordinates to check that the prescription (11.2) indeed defines a unique connection. One can already check (exercise) that such a connection would have to satisfy (1-3) above.

11.4. **Local coordinate description.** Let A be a connection and $\tau = \{e_\alpha\}$ be a local frame for E. The "connection form" of A with respect to τ is a matrix of 1-forms defined by

$$\nabla_A e_\beta = (A^\tau)^\alpha{}_\beta e_\alpha.$$

Given local coordinates $\{x^i\}$ on M, we can further write

$$(A^\tau)^\alpha{}_\beta = (A^\tau)^\alpha_{i\beta} dx^i$$

where $(A^\tau)^\alpha_{i\beta}$ are functions. In this notation,

$$\nabla_{A, \frac{\partial}{\partial x_i}} e_\beta = (A^\tau)^\alpha_{i\beta} e_\alpha.$$

Often, we will abusively omit τ from the RHS and A from the LHS.

For a section

$$s = s^\alpha e_\alpha$$

in this local frame, we have

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x_i}} s &= (ds^\alpha \otimes e_\alpha)\left(\frac{\partial}{\partial x^i}\right) + s^\alpha \nabla_{\frac{\partial}{\partial x^i}} e_\alpha \\ &= \left(\frac{\partial s^\alpha}{\partial x^i} + A^\alpha_{i\beta} s^\beta\right) e_\alpha. \end{aligned} \tag{11.3}$$

(Here I pulled a quick switcharoo between α and β in the second term.) Because of the formula (11.3), one defines

$$\nabla_i s^\alpha := (\nabla_{\frac{\partial}{\partial x^i}} s)^\alpha = \frac{\partial s^\alpha}{\partial x^i} + A^\alpha_{i\beta} s^\beta,$$

which is a perfectly good notation, as long as you don't confuse $\nabla_i s^\alpha$ with an actual derivative of the individual function s^α.

The rule (11.2) now reads:

$$\begin{aligned}\nabla_{\gamma'(t)}\tilde{\gamma}(t) = 0 \\ &= \frac{d\gamma^j}{dt}\nabla_{\frac{\partial}{\partial x^j}}\tilde{\gamma}^\alpha(t) \\ &= \frac{d\gamma^j}{dt}\left(\frac{\partial\tilde{\gamma}^\alpha}{\partial x^j} + A^\alpha_{j\beta}\tilde{\gamma}^\beta\right).\end{aligned}$$

Here we have used the linearity (11.1). Applying the chain rule to the first term, we obtain simply

$$\frac{d\tilde{\gamma}^\alpha}{dt} = -\frac{d\gamma^j}{dt}A^\alpha_{j\beta}\tilde{\gamma}^\beta. \tag{11.4}$$

This is a linear ODE system for $\tilde{\gamma}^\alpha$, so a unique solution exists over the entire interval $[a,b]$ for any choice of initial data $\tilde{\gamma}(a) = w$. Hence, we can indeed use this rule to define a connection.

Note: If one substitutes $\tilde{\gamma}(t) = \gamma'(t) \in T_{\gamma(t)}M$ in the previous equation, we recover the geodesic equation...which after all just says that the derivative of the path should be covariantly constant along itself.

Remark 11.4. Do not fear abuse of notation. All abuse of notation in differential geometry is just suppression of labels.

11.5. **Examples.**

- If $E = X \times K^r$ is the trivial bundle, then the **product connection** is defined by
$$\tilde{\gamma}_w(t) = (\gamma(t), w).$$
- Any collection of nr^2 functions
$$\{A^\alpha_{i\beta} \mid i = 1,\dots,n \text{ and } \alpha,\beta = 1,\dots,r\}$$
defines a connection on the trivial bundle. Hence, locally (and indeed globally, as we'll show), every smooth vector bundle has a vast supply of connections.
- The Möbius bundle has a flat connection given by "keeping a vector the same length." Going once around the band negates your vector. Precisely, this is defined by $A = 0$ on U_0 and $A = 0$ on U_1, but this is not a product connection because the Möbius bundle is not a trivial bundle.
- Let g be a (pseudo-)Riemannian metric on TM. The **Levi-Civita connection** is defined by
$$\Gamma^k_{ij} = \frac{1}{2}g^{k\ell}\left(\partial_i g_{\ell j} + \partial_j g_{\ell i} - \partial_\ell g_{ij}\right)$$
in each coordinate chart. This is the unique torsion-free and metric-compatible connection on TM. Here, "metric-compatible" means
$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$
"Torsion-free" means
$$\nabla_X Y - \nabla_Y X = [X,Y],$$

which only makes sense on the tangent bundle. The point is that the above formula shows that a torsion-free metric-compatible connection exists locally, but since it's unique, it's well-defined globally. This is special to metrics on the tangent bundle; a metric on a general vector bundle does not come with a preferred connection (unless the bundle is holomorphic, in which case this is a theorem of Chern that we will prove).

11.6. **Exercises.**

1. Check that the connection defined by (11.2) satisfies (1-3) in the definition of a connection, if it exists.
2. Show that the holonomy group is a group.
3. Show that given a connection A on E, one can define a pullback connection f^*A on f^*E in an obvious way.
4. Remind yourself how to derive the formula for the Levi-Civita connection, and in so-doing, prove that it exists and is unique.

12. Curvature (3/3)

12.1. **Transformation of connections.** Let $E \to M$ be a vector bundle with a connection A. Choose coordinates $\{x^i\}$ on M and frames $\{e_\alpha\}$ and $\{e'_\alpha\}$ with

$$e_\alpha = \sigma^\beta{}_\alpha e'_\beta.$$

The connection "matrices" $A^\alpha_{i\beta}$ are defined by

$$\nabla_i e_\beta = A^\alpha_{i\beta} e_\alpha.$$

Changing to the frame $\{e'_\alpha\}$, the left-hand side is

$$\begin{aligned}\nabla_i e_\beta &= \nabla_i(\sigma^\beta{}_\alpha e'_\beta)\\ &= \partial_i \sigma^\beta{}_\alpha e_\beta + \sigma^\beta{}_\alpha \nabla_i e'_\beta\\ &= (\partial_i \sigma^\gamma{}_\alpha + \sigma^\beta{}_\alpha (A')^\gamma_{i\beta})e'_\gamma.\end{aligned}$$

The right-hand side is

$$A^\alpha_{i\beta} e_\alpha = A^\alpha_{i\beta}\sigma^\gamma{}_\alpha e'_\gamma.$$

Comparing coefficients of e'_γ, and using the notation of ordinary matrix multiplication, we see that

$$\begin{aligned}&\partial_i\sigma + A'_i\sigma = \sigma A_i\\ &\implies A'_i = \sigma A_i \sigma^{-1} - \partial_i\sigma\sigma^{-1}.\end{aligned} \tag{12.1}$$

We will henceforth refer to (12.1) as the **transformation rule** for connections.

From the parable of the transformation rule is derived the moral:

A connection is not a tensor.

However, a difference of connections *is* a tensor (in particular, a section of $T^*M \otimes \mathrm{End}(E)$) because

$$\sigma(A-B) = \sigma(A) - \sigma(B) = \sigma A \sigma^{-1} - \partial_i \sigma \sigma^{-1} - \sigma B \sigma^{-1} + \partial_i \sigma \sigma^{-1} = \sigma(A-B)\sigma^{-1}.$$

Consequently, the space of all connections on E, denoted by

$$\mathscr{A}_E,$$

is an affine space modeled on the vector space

$$\Gamma(T^*M \otimes \mathrm{End} E).$$

In particular, if A and B are connections, then $\lambda A + (1-\lambda)B$ is a connection for $0 \leq \lambda \leq 1$ (exercise).

Proposition 12.1. *Every vector bundle carries a connection; indeed, a* $\Gamma(T^*M \otimes \mathrm{End} E)$*'s worth of them.*

Proof. Choose $\{U_a\}$ a trivializing cover and a partition of unity $\{\rho_a\}$. Then $E|_{U_a} \cong U_a \times K^r$, so we have the product connection A_a on U_a. Now let

$$A = \sum \rho_a A_a.$$

Near any point, this is a convex linear combination of connections, so it is a connection globally. ◻

Recall that the bundle of **gauge transformations** of E is the fiber bundle

$$\mathscr{G}_E \subset \mathrm{End}(E)$$

whose fiber at each point consists of invertible endomorphisms. Sections of $\mathscr{G}_E$ correspond to bundle automorphisms of E. If $\sigma \in \Gamma(\mathscr{G}_E)$ and A is a connection, we can pull back (or push forward) by the isomorphism σ in an obvious way; in fact, the rule is just

$$\sigma(A) = \sigma A \sigma^{-1} - d\sigma \sigma^{-1},$$

exactly as for local changes of frame. Two connections should be considered **isomorphic** if they are equal after acting by a global gauge transformation.

At the broadest level, the field of differential geometry called "gauge theory" studies the space of isomorphism classes (moduli space) of connections modulo gauge on bundles over smooth manifolds. This space is just the quotient

$$\mathscr{B}_E := \mathscr{A}_E / \Gamma(\mathscr{G}_E).$$

In fact, $\mathscr{A}_E$ is an infinite-dimensional vector space and $\Gamma(\mathscr{G}_E)$ is an infinite-dimensional Lie group, so this is a very well-behaved quotient. In particular, away from fixed points (which actually correspond to *reducible* connections), it is easy to make $\mathscr{B}_E$ into a Banach (indeed, a Hilbert) manifold.

Given how time is flying, we probably won't get to say too much more about the global shape of the space $\mathscr{B}_E$...one can read chapters 4-5 of Donaldson and Kronheiemer, or for an easier reference, the book by Freed and Uhlenbeck.

Remark 12.2. Physicists have other opinions and would call the study of $\mathscr{B}_E$ "classical gauge theory." Differential geometers would prefer not to call this "gauge theory" at all because it is only the study of vector bundles and connections modulo isomorphism, which is the most natural thing in the world.

12.2. **Curvature: Intuitive definition.** We've now defined connections and seen that vector bundles have tons of them. But many of these are isomorphic, and we want to be able to tell them apart (or in particular, tell which ones are trivial). So, as with any moduli problem, we need to extract an invariant to see what's going on.

Suppose we have a connection A on $E \to M$. Fix local coordinates $\{x^i\}$ around $p \in M$ so that p is the origin of the chart, and fix two indices i, j. Let $\gamma_{s,t}$ be the loop at p that is a coordinate rectangle of width s and height t, oriented counter-clockwise. We will compute $T^A_{\gamma_{s,t}}$ for small s and t.

Choose a gauge (=frame) $\{e_\alpha\}$ such that, in coordinates, $e_\alpha(0) = 0$. For any x in the chart, let $[0, x]$ be the line segment from the origin to x, and let

$$\gamma_x : [0,1] \to [0,x]$$

be a radial (in coordinates) path from 0 to x.

Let $e_\alpha(x) = T^A_{\gamma_x}(e_\alpha(0))$. This gives a smooth frame with $(\nabla_i e_\alpha)(0) = 0$ for all α and

$$(\nabla_i e_\alpha)(0, \ldots, x^i, \ldots, 0) = 0.$$

Then $A^\alpha_{i\beta}$ are smooth functions (since defined by varying the parameters of an ODE) and satisfy $A^\alpha_{i\beta}(0) = 0$ and $A^\alpha_{i\beta}(0, \ldots, x^i, \ldots, 0) = 0$. The Taylor expansion of A_i at 0 in the x^i, x^j-plane reads

$$\begin{aligned} A_i &= 0 + s\partial_i A_i + t\partial_j A_i + O(2) \\ &= t\partial_j A_i(0) + O(2). \end{aligned}$$

Similarly, we have

$$A_j = s\partial_i A_j(0) + O(2).$$

Now,

$$T_{\gamma_{s,t}} = T_{\gamma_4} \circ T_{\gamma_3} \circ T_{\gamma_2} \circ T_{\gamma_1}.$$

Since T_{γ_1} and T_{γ_4} are both along radial rays in our coordinates, we have $T_{\gamma_1} = \mathbf{1} = T_{\gamma_4}$. For the other ones, it's easy to see from the above Taylor expansions and the ODE (11.4) that we must have:[19]

$$\begin{aligned} T_{\gamma_2} &= \mathbf{1} - ts\partial_i A_j(0) + O(3) \\ T_{\gamma_3} &= \mathbf{1} + st\partial_j A_i(0) + O(3). \end{aligned}$$

Composing all of these, we obtain

$$T_{\gamma_{s,t}} = \mathbf{1} - st(\partial_i A_j(0) - \partial_j A_i(0)) + O(3) \tag{12.2}$$

The coefficient of $-st$ is the **curvature**, $F_{ij}(0)$, at the origin.

[19] In class I actually had the signs wrong here. It should be $-$, because the ODE for parallel transport has $-A$ on the RHS.

12.3. **Curvature: Lame definition.** The **curvature** of $X, Y \in \Gamma(TM)$ is the operator on $\Gamma(E)$ defined by

$$F(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}. \tag{12.3}$$

One can check directly using the Leibniz rule (exercise) that $F(X,Y)$ is C^∞-bilinear in X and Y, as well as C^∞-linear over $\Gamma(E)$. It therefore defines a section of the vector bundle $\Omega^2 \otimes \mathrm{End} E$. Hence, we have the well-known fact that although connections themselves are not tensors, their curvatures *are*.

Note: this also follows by considering the intuitive definition in the last subsection more carefully; or by following the slick definition of curvature that we will give in the next subsection.

Let's get a formula for the components of the tensor we've just defined, which will also allow us to compare explicitly with the definition in the previous subsection.

Convention. For $\omega \in \Omega^k$, and given local coordinates $\{x^i\}$ on M, we define its tensor components $\omega_{i_1 \cdots i_k}$ by the rule:

$$\omega = \frac{1}{k!} \omega_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where ω is assumed to be alternating. This also works for forms tensored with any bundle. In particular, for the curvature tensor, the components F_{ij} are defined by

$$F = \frac{1}{2} F_{ij} dx^i \wedge dx^j,$$

or, also choosing a frame for E, by

$$F = \frac{1}{2} F_{ij}{}^\alpha{}_\beta dx^i \wedge dx^j \otimes e_\alpha \otimes e^\beta.$$

Proposition 12.3. *We have*

$$F_{ij}{}^\alpha{}_\beta = \partial_i A^\alpha_{j\beta} - \partial_j A^\alpha_{i\beta} + A^\alpha_{i\gamma} A^\gamma_{j\beta} - A^\alpha_{j\gamma} A^\gamma_{i\beta}.$$

Proof. Applying the formula (12.3) to the commuting vector fields $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial x_j}$, and to the section e_β of E, we have

$$\begin{aligned}
F\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) e_\beta &= F_{ij}{}^\alpha{}_\beta e_\alpha \\
&= \left(\nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}}\right) e_\beta \\
&= \nabla_{\frac{\partial}{\partial x^i}} (A^\alpha_{j\beta} e_\alpha) - \nabla_{\frac{\partial}{\partial x^j}} (A^\alpha_{i\beta} e_\alpha) \\
&= \left(\frac{\partial}{\partial x^i} A^\alpha_{j\beta}\right) e_\alpha + A^\alpha_{j\beta} \nabla_{\frac{\partial}{\partial x^i}} e_\alpha - \left(\frac{\partial}{\partial x^j} A^\alpha_{i\beta}\right) e_\alpha - A^\alpha_{i\beta} \nabla_{\frac{\partial}{\partial x^j}} e_\alpha \\
&= (\partial_i A^\alpha_{j\beta} - \partial_j A^\alpha_{i\beta}) e_\alpha + (A^\alpha_{j\beta} A^\gamma_{i\alpha} - A^\alpha_{i\beta} A^\gamma_{j\alpha}) e_\gamma \\
&= (\partial_i A^\alpha_{j\beta} - \partial_j A^\alpha_{i\beta} + A^\alpha_{i\gamma} A^\gamma_{j\beta} - A^\alpha_{j\gamma} A^\gamma_{i\beta}) e_\alpha,
\end{aligned} \tag{12.4}$$

where we have exchanged α and γ in the second term. ☐

Note: Plugging in 0 in radial gauge, we see that the formula of the proposition agrees with the coefficient of $-st$ in (12.2).

12.4. **Curvature: Better definition.** We first consider: how does a connection interact with bundle operations? Suppose A is a connection on E.

- Duals: Let $\{e^\alpha\}$ be the dual frame on E^* to a given frame on E, and let $t = t_\alpha e^\alpha$ be a section of E^*. Then we define a connection (also called A) on E^* by

$$\nabla_A t = dt_\alpha \otimes e^\alpha - A^\alpha{}_\beta t_\alpha e^\beta.$$

  Given any section s of E, this satisfies

$$d(t(s)) = (\nabla_A t)(s) + t(\nabla_A s). \tag{12.5}$$

  (Exercise.)
- Tensor product: let A and B be connections on E and F. These induce a "coupled" connection on $E \otimes F$ by the prescription:

$$\nabla_{\frac{\partial}{\partial x^i}}(s^{\alpha\beta} e_\alpha \otimes f_\beta) = (\partial_i s^{\alpha\beta} + A^\alpha_{i\gamma} s^{\gamma\beta} + B^\beta_{i\gamma} s^{\alpha\gamma}) e_\alpha \otimes f_\beta.$$

  One can check (exercise) that this is a well-defined connection.
- Letting $F = E^*$ in the previous item, we have $\mathrm{End} E = E \otimes E^*$. There is a natural map

$$\mathrm{Tr} : E \otimes E^* \to \underline{K}$$

  gotten by letting a section and the dual eat each other. This is known as the **trace map**, for reasons which one can check in local coordinates (exercise). Then the coupled connection ∇ defined in the previous item also satisfies the Leibniz rule:

$$\partial_i \,\mathrm{Tr}(s \otimes t) = \mathrm{Tr}(\nabla_i s \otimes t + s \otimes \nabla_i t).$$

- We have the notation

$$\Omega^k(E) \coloneqq \Omega^k \otimes E,$$

  where one thinks of this as the bundle of k-forms "valued" in E. Define the **covariant exterior derivative**

$$D_A : \Omega^k(E) = \Omega^k \otimes E \xrightarrow{\mathbf{1} \otimes \nabla_A} \Omega^k \otimes T^*M \otimes E \xrightarrow{\wedge} \Omega^{k+1} \otimes E = \Omega^{k+1}(E). \tag{12.6}$$

  In coordinates, if

$$s = s^\alpha e_\alpha \otimes dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Gamma(\Omega^k(E)),$$

  we have

$$D_A(s) = (\nabla_i s^\alpha) e_\alpha \otimes dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \tag{12.7}$$

Definition 12.4. The **curvature operator** is given by

$$F_A = D_A^2 : E = \Omega^0(E) \to \Omega^2(E).$$

12.5. **Exercises.**

1. Show that a convex linear combination of connections is again a connection.
2. Check that the definition (12.3) gives a tensor.
3. Check explicitly that parallel transporting a basis of T_pM radially yields a *smooth* frame in a neighborhood of p, as used in the calculation of holonomy along around $\gamma_{s,t}$.
4. Check the well-definedness and stated properties for connections on duals, tensors, and End's.
5. Prove the following general commutation formula, for T a general tensor, *i.e.*, a section of $E^{\otimes k} \otimes (E^*)^{\otimes \ell}$:
$$\left[\nabla_{\frac{\partial}{\partial x^i}}, \nabla_{\frac{\partial}{\partial x^j}}\right] T^{\alpha\beta\cdots}{}_{\delta\cdots} = F_{ij}{}^{\alpha}{}_{\gamma} T^{\gamma\beta\cdots}{}_{\delta\cdots} + F_{ij}{}^{\beta}{}_{\gamma} T^{\alpha\gamma\cdots}{}_{\delta\cdots} + \cdots$$
$$- F_{ij}{}^{\gamma}{}_{\delta} T^{\alpha\beta\cdots}{}_{\gamma\cdots} - \cdots.$$

13. Flatness and curvature (3/8)

Let A be a connection. Recall that if $\{e_\alpha\}$ and $\{e'_\alpha\}$ are local frames with $e_\alpha = \sigma^\beta{}_\alpha e'_\beta$, then the transformation law says that after change of frame, A is
$$\sigma A \sigma^{-1} - d\sigma \cdot \sigma^{-1}.$$
If we also choose coordinates $\{x^i\}$ on M, we can further write
$$A = A^\alpha_{i\beta}\, dx^i \otimes e_\alpha \otimes e^\beta.$$
If $\{y^j(x^i)\}$ is a change of coordinates and A' the coordinate-changed connection form, then
$$(A')^\alpha_{j\beta} e_\alpha = \nabla_{\frac{\partial}{\partial y^j}} e_\beta = \nabla_{\left(\frac{\partial x^i}{\partial y^j}\frac{\partial}{\partial x^i}\right)} e_\beta = \frac{\partial x^i}{\partial y^j} \nabla_{\frac{\partial}{\partial x^i}} e_\beta = \frac{\partial x^i}{\partial y^j} A^\alpha_{i\beta} e_\alpha.$$
Hence
$$A'_j = \frac{\partial x^i}{\partial y^j} A_i,$$
just like for an ordinary 1-form. So although the "connection form" A is not a tensor overall, it does transform like a tensor with respect to the cotangent bundle factor (*i.e.* the "Latin" index).

Recall that curvature was defined to be
$$F = \frac{1}{2} F_{ij}{}^\alpha{}_\beta\, dx^i \wedge dx^j \otimes e_\alpha \otimes e^\beta \in \Omega^2(\mathrm{End} E),$$
where in a local frame / using matrix multiplication, we can write

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]\,. \tag{13.1}$$

Theorem 13.1. $F_A \equiv 0$ *if and only if A is flat, i.e., there exist local frames in which the connection forms vanish identically.*

Proof. Supposing that $F_A \equiv 0$, we must construct a local frame in which $A \equiv 0$.

Choose coordinates $\{x^i\}$ on M, centered at $p \in M$, and let $\{e_\alpha\}$ be a frame obtained by parallel transport from $e_\alpha(0)$ along radial rays in the chart. Choose a 2-plane through the origin in the chart, and let r and θ be polar coordinates on the plane. Over this plane, the connection form is

$$A = A^\alpha_{r\beta}\, dr \otimes e^\beta \otimes e_\alpha + A^\alpha_{\theta\beta}\, d\theta \otimes e^\beta \otimes e_\alpha.$$

Since the e_α's were constructed by radial parallel transport, if $\vec{r} = x^i \frac{\partial}{\partial x^i}$ is the radial vector field, then

$$\nabla_{\vec{r}} e_\alpha(x) \equiv 0.$$

Consequently, $A_r \equiv 0$ in this frame. Moreover, the coefficients of curvature are

$$\begin{aligned} F_{r\theta} &= \partial_r A_\theta - \partial_\theta A_r + [A_r, A_\theta] \\ &= \partial_r A_\theta, \end{aligned}$$

since $A_r \equiv 0$ hence also $\partial_\theta A_r \equiv 0$. Since $F_A \equiv 0$ by assumption, we obtain

$$\partial_r A_\theta \equiv 0.$$

But since $\frac{\partial}{\partial\theta} \to 0$ as $r \to 0$, we have $A_\theta \to 0$ as $r \to 0$. (This follows since the frame is smooth, and connections do in fact transform as tensors in the tangent-vector variable, as discussed above.) Hence $A_\theta \equiv 0$ as well.

Since the 2-plane through the origin was arbitrary, we conclude that $A \equiv 0$ in the radial frame. $\square$

Corollary 13.2. *Assume M is connected. Vector bundles $E \to M$ with flat connections, up to isomorphism, are in one to one correspondence with representations of $\pi_1(M)$, up to conjugation.*

Proof idea. Fix $x_0 \in M$. Given a flat connection A, we obtain a representation by

$$\begin{aligned} \pi_1(M, x_0) &\to \mathrm{GL}_K(E_{x_0}) \\ [\gamma] &\mapsto T^A_\gamma. \end{aligned}$$

It's an exercise to check that this is well-defined, *i.e.*, independent of the homotopy class of the loop. Then it's obvious that acting by an isomorphism of the bundle just conjugates by the action of this isomorphism on E_{x_0}.

Conversely, given a representation, one lifts the action of the group of deck transformation to the trivial bundle over $\tilde{M}$ (the universal cover) using this representation. The quotient by this action gives the required flat bundle on M.

These operations are inverses, so we have a one-to-one correspondence. $\square$

Recall our better definition of curvature (Definition 12.4), which was based on the definition of the covariant exterior derivative

$$\begin{aligned} D_A : \Omega^k(E) &\to \Omega^{k+1}(E) \\ \omega = s \otimes dx^{i_1} \wedge \cdots \wedge dx^{i_k} &\mapsto \nabla_i s \otimes dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= (\partial_i s + A_i \cdot s) \otimes dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \end{aligned}$$

As abusively as usual, we'll henceforth write

$$d\omega := \partial_i s \otimes dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$
$$A \wedge \omega := A_i \cdot s \otimes dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

so the above formula becomes simply

$$D_A \omega = d\omega + A \wedge \omega.$$

For a section s of E, we now compute

$$\begin{aligned} D_A^2 s &= D_A(ds + A \cdot s) \\ &= d(ds + As) + A \wedge (ds + As) \\ &= d^2 s + dA \cdot s - A \wedge ds + A \wedge ds + A \wedge A \cdot s \\ &= (dA + A \wedge A) \cdot s. \end{aligned} \tag{13.2}$$

This formula shows that the "curvature operator" of Definition 12.4 is in fact just multiplication by the End$E$-valued 2-form

$$F_A = dA + A \wedge A. \tag{13.3}$$

This is Cartan's formula for F_A. In particular, F_A is a manifestly well-defined C^∞-linear operator on s, as one sees direction from the formula (13.2).

Unwinding the local coordinate definition, one can see that (13.1) and (13.3) refer to the same object (exercise).

13.1. **Exercises.**

1. Fill in the details in the proof of Corollary 13.2.
2. Make sure you can see that the formulae (13.1) and (13.3) are the same.
3. Let A be a connection on E, with curvature F_A. Let A also define a connection on E^* by the rule (12.4). For t a section of E^*, show that

$$D_A^2 t = -t \cdot F_A.$$

14. Chern-Weil forms (3/10)

14.1. **Reality check: curvature is a tensor in our notation.** Let E be a vector bundle, A a connection, and fix a local frame $\tau = \{e_\alpha\}$. As explained above, we often suppress the frame label and just write $A = A^\tau$.

Cartan's formula (13.3) reads

$$F_A = dA + A \wedge A.$$

The wedge product in this expression is the following canonical bundle morphism:

$$\Omega^k(\mathrm{End}E) \otimes \Omega^\ell(\mathrm{End}E) \to \Omega^{k+\ell}(\mathrm{End}E) \tag{14.1}$$

$$(\omega^\alpha{}_\beta) \otimes (\eta^\gamma{}_\delta) \mapsto \omega^\alpha{}_\gamma \wedge \eta^\gamma{}_\delta. \tag{14.2}$$

Let's check (for the third time) that F_A transforms as a tensor for a change of frame $\sigma = \sigma^\alpha{}_\beta$. Let

$$A' = A^{\sigma(\tau)} = \sigma A^\tau \sigma^{-1} - d\sigma\sigma^{-1}.$$

Now,

$$\begin{aligned} F_{A'} = dA' + A' \wedge A' &= d(\sigma A\sigma^{-1} - d\sigma\sigma^{-1}) + (\sigma A\sigma^{-1} - d\sigma\sigma^{-1}) \wedge (\sigma A\sigma^{-1} - d\sigma\, \sigma^{-1}) \\ &= d\sigma \wedge A\sigma^{-1} + \sigma dA\sigma^{-1} - \sigma A \wedge (-\sigma^{-1} d\sigma\sigma^{-1}) \\ &- d^2\sigma\, \sigma^{-1} + d\sigma \wedge (-\sigma^{-1} d\sigma\, \sigma^{-1}) \\ &+ \sigma A\sigma^{-1} \wedge \sigma A\sigma^{-1} - d\sigma\sigma^{-1} \wedge \sigma A\sigma^{-1} \\ &- \sigma A\sigma^{-1} \wedge d\sigma\sigma^{-1} + d\sigma\sigma^{-1} \wedge d\sigma\sigma^{-1}. \end{aligned}$$

Note that $d\sigma\sigma^{-1} \wedge \sigma A\sigma^{-1} = d\sigma \wedge A\sigma^{-1}$, so we can cancel 3 pairs of terms and are left with:

$$\sigma dA\sigma^{-1} + \sigma A \wedge A\sigma^{-1} = \sigma(dA + A \wedge A)\sigma^{-1} = \sigma F_A \sigma^{-1}.$$

14.2. **Identities for curvature.** Above, we defined the covariant exterior derivative

$$\begin{aligned} D_A : \Omega^k(E) &\to \Omega^{k+1}(E) \\ \alpha &\mapsto d\alpha + A \wedge \alpha. \end{aligned}$$

Replacing E by $\mathrm{End}E$, we have

$$D_A : \Omega^k(\mathrm{End}E) \to \Omega^{k+1}(\mathrm{End}E),$$

given by

$$(D_A\omega)^\alpha{}_\beta = (d\omega)^\alpha{}_\beta + A^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta - A^\gamma{}_\beta \wedge \omega^\alpha{}_\gamma.$$

This just reexpresses the definition of D_A, using the connection induced by A on tensors and duals (see §12.4). Using the above $\wedge$ operation on $\mathrm{End}E$-valued forms, we can rewrite this as:

$$D_A\omega = d\omega + A \wedge \omega + (-1)^{k+1} \omega \wedge A. \tag{14.3}$$

Proposition 14.1. *The operator D_A on $\mathrm{End}E$ obeys the following identities:*

(1) $D_A(\omega \wedge \eta) = D_A\omega \wedge \eta + (-1)^k \omega \wedge D_A\eta$, *where* $\omega \in \Omega^k(\mathrm{End}E)$ *and* $\eta \in \Omega^\ell(\mathrm{End}E)$.
(2) $d\,\mathrm{Tr}\, \omega \wedge \eta = \mathrm{Tr}(D_A\omega \wedge \eta + (-1)^k \omega \wedge D_A\eta)$
(3) $d\,\mathrm{Tr}\, \omega \wedge \eta \wedge \mu = \mathrm{Tr}(D_A\omega \wedge \eta \wedge \mu + (-1)^k \omega \wedge D_A\eta \wedge \mu + (-1)^{k+\ell} \omega \wedge \eta \wedge D_A\mu)$, *where* ω, η, *and* μ *are* k-, ℓ-, *and* m- *forms, respectively. Similar identities hold for general wedge products.*
(4) $D_A^2\omega = F_A \wedge \omega + (-1)^{k+1} \omega \wedge F_A$.

Proof. Exercises. You can either do these directly using the formula (14.3), or by using either of the definitions (12.6) or (12.7) and properties of ∇_A. □

Theorem 14.2 ("Second" Bianchi identity)**.**

$$D_A F_A = 0.$$

Proof 1 of Theorem 14.2.

$$\begin{aligned} D_A F_A &= d(dA + A \wedge A) + A \wedge (dA + A \wedge A) - (dA + A \wedge A) \wedge A. \\ &= d^2 A + dA \wedge A - A \wedge dA + A \wedge dA + A \wedge A \wedge A - dA \wedge A - A \wedge A \wedge A = 0. \end{aligned}$$

□

Proof 2 of Theorem 14.2. In different notation, we have

$$\begin{aligned} D_A F_A &= \nabla_{\frac{\partial}{\partial x^i}} \left(\frac{1}{2} F\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right) \right) dx^i \wedge dx^j \wedge dx^k \\ &= \frac{1}{12} \begin{pmatrix} \nabla_{\frac{\partial}{\partial x^i}} F\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right) + \nabla_{\frac{\partial}{\partial x^j}} F\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right) + \nabla_{\frac{\partial}{\partial x^k}} F\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \\ - \nabla_{\frac{\partial}{\partial x^i}} F\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j} \right) - \nabla_{\frac{\partial}{\partial x^j}} F\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^i} \right) - \nabla_{\frac{\partial}{\partial x^k}} F\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i} \right) \end{pmatrix} dx^i \wedge dx^j \wedge dx^k, \end{aligned}$$

where we have used the antisymmetry of $dx^i \wedge dx^j \wedge dx^k$. (We could also have arrived at this expression directly from (12.6).) Since coordinate vector fields commute, the expression in parentheses is equal to

$$\left[\nabla_{\frac{\partial}{\partial x^i}}, [\nabla_{\frac{\partial}{\partial x^j}}, \nabla_{\frac{\partial}{\partial x^k}}] \right] + \left[\nabla_{\frac{\partial}{\partial x^j}}, [\nabla_{\frac{\partial}{\partial x^k}}, \nabla_{\frac{\partial}{\partial x^i}}] \right] + \left[\nabla_{\frac{\partial}{\partial x^k}}, [\nabla_{\frac{\partial}{\partial x^i}}, \nabla_{\frac{\partial}{\partial x^j}}] \right].$$

But according to the Jacobi identity, which holds for any triple of linear operators, this must vanish. □

Remark 14.3. The so-called "first" Bianchi identity $R_{ijk\ell} + R_{ik\ell j} + R_{i\ell jk} = 0$ only applies to the Riemann curvature tensor (*i.e.* the curvature of the Levi-Civita connection associated to a metric on TM), and does not even make sense for a connection on a general vector bundle. For this reason, when discussing general vector bundles and connections, it's not ambiguous to simply call Theorem 14.2 the "Bianchi identity."

Lemma 14.4. *If* $a \in \Omega^1(\mathrm{End} E)$*, then*

$$F_{A+a} = F_A + D_A a + a \wedge a.$$

In particular, if A_t *is a family of connections, then*

$$\frac{d}{dt} F_{A_t} = D_{A_t} \left(\frac{dA_t}{dt} \right).$$

Proof. We calculate

$$\begin{aligned} F_{A+a} &= d(A + a) + (A + a) \wedge (A + a) = dA + da + A \wedge A + A \wedge a + a \wedge A + a \wedge a \\ &= F_A + D_A a + a \wedge a. \end{aligned}$$

Here we have used (14.3), with $k = 1$.

The second identity is the infinitesimal version of the first. □

14.3. **Chern-Weil forms.**

Theorem 14.5 (Chern-Weil)**.** *For each $p \in \mathbb{N}$, the $2p$-form*

$$\mathrm{Tr}_K(\overbrace{F_A \wedge \cdots \wedge F_A}^{p \text{ times}})$$

is closed, and its cohomology class is independent of the connection A on E.

Proof. Applying the Leibniz rule (3) of Proposition 14.1, we have

$$d\,\mathrm{Tr}(F_A \wedge \cdots \wedge F_A) = \mathrm{Tr}(D_A F_A \wedge \cdots \wedge F_A) + \mathrm{Tr}(F_A \wedge D_A F_A \wedge \cdots \wedge F_A) + \cdots.$$

By Theorem 14.2, each term vanishes.

Next, let A and B be connections on E, and put

$$a = B - A \in \Omega^1(\mathrm{End} E).$$

Let $A_t \coloneqq A + ta$. Then

$$\begin{aligned}
\frac{d}{dt}\mathrm{Tr}(F_{A_t} \wedge \cdots \wedge F_{A_t}) &= \mathrm{Tr}(\dot{F}_{A_t} \wedge F_{A_t} \wedge \cdots \wedge F_{A_t} + F_{A_T} \wedge \dot{F}_{A_T} \wedge \cdots \wedge F_{A_t}) + \cdots) \\
&= \mathrm{Tr}(D_A a \wedge F_{A_t} \wedge \cdots \wedge F_{A_t} + F_{A_t} \wedge D_A a \wedge \cdots \wedge F_{A_t} + \cdots) \\
&= p\,\mathrm{Tr}(D_A a \wedge F_{A_t} \wedge \cdots \wedge F_{A_t}) \\
&= p\,d\,\mathrm{Tr}(a \wedge F_{A_t} \wedge \cdots F_{A_t}),
\end{aligned}$$

where we have used the cyclic property of the trace, the Leibniz rule (3), and the Bianchi identity. Since $A_0 = A$ and $A_1 = B$, we obtain

$$\begin{aligned}
\mathrm{Tr}\, F_B \wedge \cdots \wedge F_B - \mathrm{Tr}(F_A \wedge \cdots \wedge F_A) &= \int_0^1 d(p\,\mathrm{Tr}(a \wedge F_{A_t} \wedge \cdots \wedge F_{A_t}))\,dt \\
&= d\left[\int_0^1 p\,\mathrm{Tr}(a \wedge F_{A_t} \wedge \cdots F_{A_t})\,dt\right].
\end{aligned}$$

This shows that

$$[\mathrm{Tr}\, F_A \wedge \cdots \wedge F_A] = [\mathrm{Tr}(F_B \wedge \cdots \wedge F_B)] \in H^{2p}(M),$$

as claimed. □

Corollary 14.6. $[\mathrm{Tr}(F_A \wedge \cdots \wedge F_A)] \in H^{2p}(M)$ *defines a characteristic class.*

Proof. By the previous Theorem, this expression is a well-defined function of the bundle E up to isomorphism. We need only check that it behaves properly under pullback. This is true because $f^* F_A = F_{f^* A}$, which is clear from (13.3), so

$$f^*\,\mathrm{Tr}(F_A \wedge \cdots \wedge F_A) = \mathrm{Tr}(F_{f^*A} \wedge \cdots \wedge F_{f^*A}).$$

□

Definition/Lemma 14.7. *Let E be a vector bundle over $\mathbb{C}$. The first Chern class is given by*

$$c_1(E) = \left[\frac{i}{2\pi}\mathrm{Tr}_{\mathbb{C}}\, F_A\right] \in H^2(M).$$

This agrees with Definition 7.7/8.5 of c_1.

Proof. It suffices to check that $c_1(\mathscr{O}(-1) \to \mathbb{CP}^1) = -x$ (exercise: think about why this is sufficient). Here, x is the generator of $H^2(\mathbb{CP}^1, \mathbb{Z}) \cong \mathbb{Z}$ that pairs to one with the fundamental class.

Let $U_0 = (\{z\}, \tau_0)$ and $U_1 = (\{w\}, \tau_1)$, with the frames τ_0, τ_1 given in Example 6.1.3. above. We have $z = \frac{1}{w}$ on the overlap. We showed that $\tau_0 = z\tau_1$, so the transition function is just

$$\sigma = \sigma_{01}(z) = z.$$

Let

$$A^{\tau_0} = \frac{dz\bar{z}}{1+|z|^2}$$

on U_0, and

$$A^{\tau_1} = -\frac{dw\bar{w}}{1+|w|^2}$$

on U_1.

Claim. This gives a well-defined connection on $\mathscr{O}(-1) \to \mathbb{CP}^1$.

Proof of claim.

$$\begin{aligned}
\sigma A^{\tau_0}\sigma^{-1} - d\sigma\,\sigma^{-1} &= zA^{\tau_0}z^{-1} - dz\,z^{-1} \\
&= A^{\tau_0} - dz\,z^{-1} \\
&= dz\bar{z}\left(\frac{1}{1+|z|^2} - \frac{1}{|z|^2}\right) \\
&= \frac{dz\bar{z}(|z|^2 - (1+|z|^2))}{|z|^2(1+|z|^2)} \\
&= -\frac{dz\bar{z}}{|z|^2(1+|z|^2)} \\
&= -\frac{1}{w^2}\frac{dw|w|^2}{\bar{w}(1+1/|w|^2)} \\
&= -\frac{dw\bar{w}}{1+|w|^2} = A^{\tau_1}.
\end{aligned}$$

◇

Now, we compute the curvature in the U_0 chart:

$$\begin{aligned}
F_A\big|_{U_0} &= d\left(A^{\tau_0}\right) + \underbrace{A^{\tau_0} \wedge A^{\tau_0}}_{0} \\
&= \frac{-dz \wedge d\bar{z}}{1+|z|^2} + \frac{dz\bar{z} \wedge \left(dz\bar{z} + zd\bar{z}\right)}{(1+|z|^2)^2} \\
&= \frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}\left(-(1+|z|^2) + |z|^2\right) \\
&= -\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}.
\end{aligned}$$

Here, $A^{\tau_0} \wedge A^{\tau_0} = 0$ since this is a $\mathbb{C}$-valued 1-form.

Now, $dz \wedge d\bar{z} = -2i dx \wedge dy$, so

$$\begin{aligned}\left(c_1(\mathscr{O}(-1)), \left[\mathbb{CP}^1\right]\right) &= \frac{i}{2\pi} \int_{\mathbb{C}=U_0} \frac{2i dx \wedge dy}{(1+|z|^2)^2} \\ &= -\frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \frac{r\, dr\, d\theta}{(1+r^2)^2} \\ &= -\frac{2\pi}{\pi} \int_0^{\infty} \frac{r\, dr}{(1+r^2)^2} \\ &= -\int_0^{\infty} \frac{du}{(1+u)^2} \\ &= -\left(-\frac{1}{1+u}\Big|_0^{\infty}\right) = -1.\end{aligned}$$

□

14.4. **Exercises.**

1. Prove Proposition 14.1.
2. Think about why it suffices to check $\left(c_1(\mathscr{O}(-1)), \left[\mathbb{CP}^1\right]\right) = -1$ in the above Definition/Lemma.
3. Let $\mathfrak{g}_E = \Omega^0(\mathrm{End} E)$. Given $u \in \Gamma(\mathfrak{g}_E)$, let $\sigma_t = \exp(tu) \in \Gamma(\mathscr{G}_E)$. Show that
$$\frac{d}{dt}(\sigma_{-t}(A))\Big|_{t=0} = D_A u.$$
Here, $\sigma_t(A)$ is the transformed connection by the usual rule (12.1).
4. Let A, B be connections and put $a = B - A$. Prove the following identity:
$$\mathrm{Tr}\left(F_B \wedge F_B - F_A \wedge F_A\right) = d\,\mathrm{Tr}\left(D_A a \wedge a + \frac{2}{3} a \wedge a \wedge a\right).$$
The form inside the d on the RHS is called the (relative) *Chern-Simons form.*

15. Structure group for connections, Gauss-Bonnet, higher Chern classes (3/22)

Recall: if $\{e_\alpha\}$ is a frame, then a connection A is on sufficiently small opens U given by

$$A = A^\alpha_{i\beta} dx^i \otimes e_\alpha \otimes e^\beta \in \Gamma(U, \Omega^1(\mathrm{End} E)),$$

but these local representatives do not glue to a global section. The curvature operator is locally

$$F_A = dA + A \wedge A = dA + \frac{1}{2}[A_i, A_j] dx^i \wedge dx^j \in \Omega^2(\mathrm{End} E),$$

and these do glue to an honest global section of $\Omega^2(\mathrm{End} E)$. We proved earlier that $F_A \equiv 0$ iff $A \overset{\text{loc}}{\equiv} 0$.

15.1. **Structure group for connections and metric compatibility.** Suppose that E has structure group $G \subset \mathrm{GL}(r, K)$. G acts on $\mathfrak{g} = \mathrm{Lie(G)} \subset M^{r\times r}(K)$ via the adjoint representation, so $\mathfrak{g} \subset K^r \otimes (K^r)^*$ is G-invariant. Hence, there is a well-defined subbundle $\mathfrak{g}_E \subset \mathrm{End}E = E \otimes E^*$ given by

$$(\mathfrak{g}_E)_x = \mathfrak{g} \subset \mathrm{End}E_x.$$

Equivalently, letting $P = \mathrm{Fr}_{\mathrm{G}}(E)$ be the principal G-frame bundle (per §10.4), we just have

$$\mathfrak{g}_E = \mathfrak{g}_P = P \times_{\mathrm{Ad}} \mathfrak{g}.$$

Definition 15.1. A connection on E has **structure group** G iff in each local frame, the connection forms belong to $\Omega^1(\mathfrak{g})$. In particular, the curvature F_A will be a genuine section of $\Omega^2(\mathfrak{g}_E)$.

Examples 15.2.

- Let $G = \mathrm{GL}(n, \mathbb{C}) \subset \mathrm{GL}(2n, \mathbb{R})$, then $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}) = M^{n\times n}(\mathbb{C})$, so A_i is a complex-linear matrix.
- Let $G = \mathrm{O}(n) \subset \mathrm{GL}(n, \mathbb{R})$, so

$$\mathfrak{g} = \mathfrak{o}(n) = \{\text{skew-symmetric matrices}\}.$$

  An $\mathrm{O}(n)$-structure is equivalent to a metric h on E. Then A has structure group $\mathrm{O}(n)$ iff $A^\alpha_{i\beta}$ is skew-symmetric in $\alpha \leftrightarrow \beta$. Equivalently, ∇_A must be **compatible** with h; that is,

$$d\langle s, t\rangle = \langle \nabla_A s, t\rangle + \langle s, \nabla_A t\rangle.$$

  To see the equivalence, let $\{e_\alpha\}$ be an orthonormal frame (i.e. $h_{\alpha\beta} = \delta_{\alpha\beta}$) and let $s = s^\alpha e_\alpha$ and $t = t^\alpha e_\alpha$ be sections. Then

$$\begin{aligned}
\partial_i \langle s, t\rangle = \partial_i(s^\alpha t^\alpha) &= (\partial_i s^\alpha)t^\alpha + s^\alpha(\partial_i t^\alpha) + (A^\alpha_{i\beta}s^\beta t^\alpha - A^\alpha_{i\beta}s^\beta t^\alpha)\\
&= (\partial_i s^\alpha)t^\alpha + s^\alpha(\partial_i t^\alpha) + A^\alpha_{i\beta}s^\beta t^\alpha + s^\beta A^\beta_{i\alpha}t^\alpha\\
&= (\nabla_i s^\alpha)t^\alpha + s^\alpha(\nabla_i t^\alpha)\\
&= \langle \nabla_i s, t\rangle + \langle s, \nabla_i t\rangle.
\end{aligned}$$

- Let $G = U(n) \subset \mathrm{GL}(n, \mathbb{C})$ and E is rank n over $\mathbb{C}$. Equivalently, E has an Hermitian metric $h = \langle\cdot,\cdot\rangle$. Then A has structure group $U(n)$ iff locally

$$A_i \in \mathfrak{u}(n) = \{A^\dagger = -A\}.$$

  This is again equivalent to the compatibility (Leibniz) rule:

$$d\langle s, t\rangle = \langle \nabla_A s, t\rangle + \langle s, \nabla_A t\rangle.$$

- For $G = \mathrm{SU}(n)$, we have

$$\mathfrak{g} = \mathfrak{su}(n) = \{A^\dagger = -A, \mathrm{Tr}_{\mathbb{C}} A = 0\}.$$

- Let $G = \mathrm{SU}(2)$, which we view as the unit sphere inside the quaternions $\mathbb{H}$. (See §6.2.) Then E is a G-bundle iff it is an $\mathbb{H}$-line bundle with metric. We have

$$\mathfrak{g} = \mathfrak{su}(2) = \mathrm{Im}\,\mathbb{H} = \left\langle q^1, q^2, q^3\right\rangle \cong \mathbb{R}^3.$$

One can see by examining the formula (6.7) that these indeed span the algebra of traceless skew-hermitian 2×2 matrices. Hence, the connection matrices of a connection with structure group $\mathrm{SU}(2)$ are $\mathrm{Im}\,\mathbb{H}$-valued 1-forms.

15.2. **Properties of the first Chern class, revisited.** Recall from last time that

$$[\mathrm{Tr}_K(\wedge^p F_A)] \in H^{2p}(M)$$

defines a characteristic class. Briefly, this is because

(1) $D_A F_A = 0$ by Bianchi, which, together with the Leibniz rule, implies that the above is closed.
(2) For a family A_t of connections, we have

$$\frac{d}{dt} F_{A_t} = D_{A_t}\left(\frac{\partial}{\partial t} A_t\right).$$

Combined with the Leibniz rule and Bianchi identity, this gives us

$$\begin{aligned}\frac{d}{dt}\,\mathrm{Tr}\wedge^p F_{A_t} &= p\,\mathrm{Tr}\left(\frac{d}{dt} F_{A_t} \wedge \cdots \wedge F_{A_t}\right)\\ &= p\,\mathrm{Tr}\left(\left(D_{A_t}\frac{d}{dt} A_t\right) \wedge \cdots \wedge F_{A_t}\right)\\ &= d\left(p\,\mathrm{Tr}\left(\frac{d}{dt} A_t \wedge F_{A_t} \wedge \cdots \wedge F_{A_t}\right)\right).\end{aligned}$$

Hence, writing schematically, we have

$$\frac{d}{dt}\left[\mathrm{Tr}\wedge^p F_{A_t}\right] = 0 \in H^{2p}(M),$$

which is to say, the cohomology class is constant along the path. Applying this to the straight line path between two connections, we have the independence.
(3) Since $f^* F_A = F_{f^* A}$, we have the required functoriality under pullback.

Example 15.3. For $K = \mathbb{C}$, we have $c_1(E) = \frac{i}{2\pi}[\mathrm{Tr}_{\mathbb{C}} F_A]$. Indeed, we checked last time that $c_1(\mathscr{O}_{\mathbb{CP}^1}(-1)) = -[\mathbb{CP}^1]^* = -x$.

Proposition 15.4. *For complex line bundles E and F, we have:*

(1) $c_1(E^*) = -c_1(E)$
(2) $c_1(E \oplus F) = c_1(E) + c_1(F)$
(3) $c_1(E \otimes F) = c_1(E)\mathrm{rk}(F) + \mathrm{rk}(E)c_1(F)$.

Proof. We went through great pains to prove special cases of this in Lemma 8.4 and Proposition 8.7 above; now each follows just by letting A induce a connection according to the rules in §12.4, as follows.

(1) A induces a connection on E^* by "$-A^T$" locally. The curvature of A on E^* is given by $-F_A^T$ (see Exercise 13.13.).
(2) Let A be a connection on E and B a connection on F. Then "$A \oplus B$" defines a connection on $E \oplus F$, and $F_{A \oplus B} = F_A \oplus F_B$.
(3) $A \otimes B = A \otimes \mathbf{1}_E + \mathbf{1}_F \otimes B$ and $F_{A \otimes B} = F_A \otimes \mathbf{1}_F + \mathbf{1}_E \otimes F_B$.

□

15.3. **First Chern class of surfaces.**

Definition 15.5. Let Σ be an oriented 2-manifold. Since Σ is oriented, $T\Sigma$ has structure group $\mathrm{GL}^+(2,\mathbb{R})$. We can always reduce to the orthogonal group, so we can reduce to $\mathrm{SO}(2)$, which is equal to $U(1)$. Hence, $T\Sigma$ reduces (uniquely up to isomorphism) to structure group $U(1)$, and we write

$$c_1(\Sigma) \coloneqq c_1(T\Sigma).$$

The definition makes sense more generally for any almost-complex manifold (*i.e.* a manifold whose tangent bundle is endowed with the structure of a complex vector bundle).

Example 15.6.

- $c_1(\mathbb{CP}^1) = c_1(\mathscr{O}(2)) = 2$ (*i.e.* $\left(c_1(\mathscr{O}(2)), [\mathbb{CP}^1]\right) = 2$).
- $c_1(T^2) = 0$ because the torus is parallelizable, *i.e.*, has trivial tangent bundle.

Definition 15.7. Let h be a Riemannian metric on $T\Sigma$, where Σ is a surface. We can define an almost-complex structure I on $T\Sigma$ which rotates each tangent plane 90 degrees counterclockwise. Let Γ_h be the Levi-Civita connection of h, which commutes with I, so can be viewed as a connection on the complex line bundle TX. Its curvature is a 1×1 skew-hermitian-matrix-valued 2-form, *i.e.* a purely imaginary 2-form. Hence we may define the **Gauss curvature** $K_h \in C^\infty(X,\mathbb{R})$ by the prescription

$$i\,\mathrm{Tr}_{\mathbb{C}}\, F_{\Gamma_h} = K_h\, dV_h.$$

Theorem 15.8 (Gauss-Bonnet)**.** *Let Σ_g be a compact orientable surface of genus g and h any Riemannian metric on Σ_g. Then*

$$c_1(\Sigma_g) = \frac{1}{2\pi}\int K_h\, dV_h = \chi(\Sigma_g) = 2-2g.$$

Proof. The first and third equalities are by definition; the middle equality remains.

We already know that the middle equality is true when $g = 0,1$ by the previous examples, so we may proceed by induction. As $c_1(\Sigma_g)$ is independent of the metric h, we can deform h as we wish without changing the integral of the Gauss curvature. Hence, we can decompose Σ_{g+1} (in a $C^{1,1}$ fashion) as: Σ_g, minus two round hemispheres, plus half a torus, yielding

$$\text{“}\Sigma_{g+1} = \Sigma_g - \Sigma_0 + \frac{1}{2}\Sigma_1.\text{”}$$

A gorgeous picture was drawn in class. Hence,

$$\int_{\Sigma_{g+1}} K_h dV_h = \int_{\Sigma_g} K_h dV_h - \int_{\Sigma_0} K_h dV_h + 0 = \chi(\Sigma_g) - \chi(\Sigma_0) = \chi(\Sigma_g) - 2 = 2-2(g+1).$$

By induction, we are done. □

15.4. **General method to construct characteristic classes of G-bundles.**

Let $P(X)$ be an $\mathrm{Ad}\,G$-invariant polynomial on $\mathfrak{g}$; that is, an element of

$$\left(\bigoplus_{k\geq 0}\mathrm{Sym}^k\,\mathfrak{g}^*\right)^{\mathrm{Ad}\,G}.$$

Example 15.9. Let $G = \mathrm{GL}(r, \mathbb{C})$, so $\mathfrak{g} = \mathfrak{gl}(r, \mathbb{C})$ consists of all $r \times r$ complex matrices, which we write as $X = (X^\alpha{}_\beta)$. A polynomial on $\mathfrak{g}$ is just a polynomial in the variables $X^\alpha{}_\beta$. The following are Ad-invariant:

- $\mathrm{Tr}_{\mathbb{C}}(X) = X^\alpha{}_\alpha \in \mathfrak{g}^* = \mathrm{Hom}(\mathfrak{g}, \mathbb{C})$. By the "cyclic property of the trace," we have
$$\mathrm{Tr}\, YXY^{-1} = \mathrm{Tr}\, XY^{-1}Y = \mathrm{Tr}\, X,$$
so this is indeed Ad-invariant.
- $\mathrm{Tr}_{\mathbb{C}}(X^k) = X^{\alpha_1}{}_{\alpha_2} X^{\alpha_2}{}_{\alpha_3} \cdots X^{\alpha_k}{}_{\alpha_1} \in \mathrm{Hom}(\mathrm{Sym}^k \mathfrak{g}, \mathbb{C})$.
- $\det_{\mathbb{C}}(X) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) X^1_{\sigma(1)} X^2_{\sigma(2)} \cdots X^n_{\sigma(n)}$. This satisfies
$$\det YXY^{-1} = \det X,$$
as you may know.

In particular, given any Lie subgroup $H \subset \mathrm{GL}(n, \mathbb{C})$, these each restrict to $\mathrm{Ad}\, H$-invariant polynomials on $\mathfrak{h} \subset \mathfrak{gl}(n, \mathbb{C})$. However, smaller algebras may have additional invariant polynomials.

Let $P(X)$ be an $\mathrm{Ad}\, G$-invariant polynomial and A a connection on E with structure group G. We can evaluate P on the curvature tensor, to obtain a section
$$P(F_A) \in \Omega^{*, even}(M).$$
This requires some explanation. In a local frame, the curvature is just a $\mathfrak{g}$-valued 2-form. Since 2-forms commute under the wedge product (indeed, $\Omega^{*, even}$ is a commutative algebra), it makes sense to plug a collection of 2-forms into a multivariate polynomial. Furthermore, $P(F_A)$ is globally well-defined, because
$$P(\sigma \cdot F_A \cdot \sigma^{-1}) = P(F_A)$$
by Ad-invariance.

Example 15.10. For $P(X) = \mathrm{Tr}(X^k)$, we have
$$P(F_A) = \mathrm{Tr}(F_A \wedge \cdots \wedge F_A).$$
This is just the expression that we studied in the last section, which we showed yields a well-defined characteristic class. In fact this is true for any invariant polynomial: for a general proof, see Milnor and Stasheff, "Fundamental Lemma" on p. 296.

Definition 15.11. Given any complex vector bundle E of rank r over a smooth manifold M, the **Chern classes** $c_k(E) \in H^{2k}(M)$, for $k = 0, \ldots, r$, are defined by the prescription
$$\left[\det\left(\mathbf{1} + \frac{i}{2\pi} F_A\right)\right] = \sum_{k=0}^{r} c_k(E).$$
Here, we are letting $P(X) = \det\left(\mathbf{1} + \frac{i}{2\pi} X\right)$, which is an Ad-invariant polynomial on $\mathfrak{gl}(r, \mathbb{C})$, and evaluating on F_A as discussed above. The brackets on the LHS mean that we are taking cohomology classes. We will discuss this formula more next time.

15.5. Exercises.

1. Justify the item in Example 15.2 stating that a connection on a complex line bundle is compatible with a Hermitian metric if and only if its connection forms are skew-Hermitian (*i.e.* it has structure group $\mathfrak{u}(n)$).
2. Make sure you can unpack the notation used in the proof of Proposition 15.4.
3. Check that the Gauss curvature of Definition 15.7 agrees with the usual definition (*i.e.* the sectional curvature in dimension two).

16. Whitney product formula, Pontryagin and Euler classes (3/24)

16.1. **Invariant polynomials, again.** Last time we saw that to construct a characteristic class for G-bundles, we should take

$$P(X) \in \bigoplus_{k\geq 0} \left(\mathrm{Sym}^k \mathfrak{g}^*\right)^{\mathrm{Ad}\, G}$$

an $\mathrm{Ad}\, G$-invariant polynomial on $\mathfrak{g} = \mathrm{Lie}(G)$.

Example 16.1. Let $G = \mathrm{GL}(n,\mathbb{C})$, so $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$. Write $X = (X^\alpha{}_\beta) \in \mathfrak{gl}(n,\mathbb{C})$. We previously saw that determinant, trace, and power symmetric polynomials are all $\mathrm{Ad}\, G$-invariant. The **elementary symmetric polynomials** σ_i, defined by

$$\det(\mathbf{1} + tX) = \sum_{k=0}^{n} t^k \sigma_k(X), \tag{16.1}$$

are another important example. If X is the diagonal matrix with diagonal entries $\lambda_1, \dots, \lambda_n$, then

$$\begin{aligned}
\sigma_0(X) &= 1\\
\sigma_1(X) &= \sum_i \lambda_i\\
\sigma_2(X) &= \sum_{i<j} \lambda_i \lambda_j\\
&\vdots\\
\sigma_k(X) &= \sum_{i_1<i_2<\cdots<i_k} \lambda_{i_1}\cdots\lambda_{i_k}.
\end{aligned}$$

Lemma 16.2. *Let $K = \mathbb{R}$ or $\mathbb{C}$. An $\mathrm{Ad}\, G$-invariant polynomial $P(X)$ on $\mathfrak{gl}(n,K)$ is equal to a symmetric polynomial in the eigenvalues of X.*

Proof. Any matrix with distinct eigenvalues is diagonalizable over $\mathbb{C}$, so diagonalizable matrices are dense in $\mathfrak{gl}(n)$. Hence, P is determined by its values on diagonalizable matrices, and must be a polynomial in the eigenvalues of X. Since P is invariant under $S_n \subset \mathrm{GL}(n)$, acting on the subset of diagonal matrices, it must be symmetric in the eigenvalues. □

Lemma 16.3. *If $K = \mathbb{R}$ or $\mathbb{C}$, then the elementary symmetric polynomials (resp. power symmetric polynomials) are algebraically independent generators for the ring of symmetric* *polynomials over K. Equivalently, any symmetric polynomial can be written uniquely as a polynomial in the elementary symmetric polynomials (resp. power symmetric polynomials).*

Proof. This is covered in undergrad algebra. You can also prove it first for power functions, which is easier, then use Newton's Identities. □

Corollary 16.4. *The elementary symmetric polynomials are expressible as polynomials in the power symmetric polynomials, and vice-versa.*

Example 16.5.

- $\sigma_1(X) = \lambda_1 + \cdots + \lambda_n = \operatorname{Tr} X = p_1(X)$
- $\sigma_2(X) = \sum_{i<j} \lambda_i \lambda_j = \frac{1}{2}\left(\left(\sum_i \lambda_i\right)^2 - \sum_i \lambda_i^2\right) = \frac{1}{2}(p_1(X)^2 - p_2(X))$.

16.2. **The Chern-Weil homomorphism.** Given any Lie subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$ with Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C})$, let $P(X)$ be an $\operatorname{Ad} G$-invariant polynomial on $\mathfrak{g}$ and let A be a connection on a vector bundle E with structure group G. We can evaluate

$$P(F_A) \in \Omega^{*,even}(M)$$

as follows. Locally,

$$F = F^\alpha{}_\beta e_\alpha \otimes e^\beta$$

with $F^\alpha{}_\beta \in \Omega^2$, a complex-valued 2-form, so

$$P(F_A) = P(F^\alpha{}_\beta).$$

This makes sense because even forms commute. Moreover, the form is independent of the frame because P is $\operatorname{Ad} G$-invariant and F_A transforms as a tensor for $\operatorname{Ad} G$.

Lemma 16.6. *$P(F_A)$ is closed and independent of A on E, so $c_P(E) = [P(F_A)]$ defines a characteristic class for G-bundles.*

Proof. This is called the "Fundamental Lemma" in Milnor-Stasheff, Appendix C. For the case $G = \mathrm{GL}(n, \mathbb{C})$, we can argue as follows. Last time we proved that $p_k(X) = \operatorname{Tr}(X^k)$ is Ad-invariant, hence $[p_k(F_A)]$ defines a characteristic class. Since power symmetric polynomials generate all symmetric polynomials, it follows that all symmetric polynomials on $\mathfrak{gl}(n, \mathbb{C})$ give characteristic classes. □

Recall that characteristic classes for G-bundles are equivalent to cohomology classes in $H^*(BG)$. Hence, we can define the **Chern-Weil homomorphism**

$$\begin{aligned} (\mathrm{Sym}^\bullet \mathfrak{g}^*)^{\operatorname{Ad} G} &\to H^{*,even}(BG, \mathbb{C}) \\ P &\mapsto c_P(EG). \end{aligned}$$

Theorem 16.7 (Chern)**.** *For a semisimple structure group G, the Chern-Weil homomorphism is an isomorphism.*

We will not discuss the proof except in the case $G = \mathrm{GL}(n, \mathbb{C})$, where this will follow from Theorem 16.12 below.

16.3. **Chern classes, again.** For $k = 0, \dots, r = \mathrm{rk}_{\mathbb{C}}(E)$, the **$k$'th Chern class** of a complex bundle is given by

$$c_k(E) := \left[\sigma_k\left(\frac{i}{2\pi}F_A\right)\right] \in H^{2k}(M).$$

By the construction (16.1) of the elementary symmetric polynomials, this is equivalent to Definition 15.11 from last class.

Example 16.8. Recall from Example 16.5 that $\sigma_2(X) = \frac{1}{2}(p_1(X)^2 - p_2(X))$. Hence

$$c_2(E) = \frac{1}{2}\left(\frac{i}{2\pi}\right)^2\left[(\mathrm{Tr}\,F_A)^2 - \mathrm{Tr}(F_A \wedge F_A)\right] = \frac{1}{8\pi^2}\left[\mathrm{Tr}\,F_A \wedge F_A\right] + \frac{1}{2}c_1(E)^2.$$

Expressions for $c_3, c_4, \dots$ in terms of the $\mathrm{Tr}(F_A \wedge \dots \wedge F_A)$ can be determined in similar fashion, just by calculating with symmetric polynomials!

Definition 16.9. The **total Chern class** is given by

$$c(E) := \sum_{k=0}^{r} c_k(E) = \left[\det\left(\mathbf{1} + \frac{i}{2\pi}F_A\right)\right].$$

Theorem 16.10 (Whitney product formula)**.** *For a direct sum of two complex vector bundles E and F, the total Chern class is given by*

$$c(E \oplus F) = c(E) \smile c(F).$$

Proof. By definition, we have

$$\begin{aligned}\det\left(\mathbf{1}_{E\oplus F} + \frac{i}{2\pi}(F_A \oplus F_B)\right) &= \det\begin{pmatrix}\mathbf{1}_E + \frac{i}{2\pi}F_A & 0 \\ 0 & \mathbf{1}_F + \frac{i}{2\pi}F_B\end{pmatrix} \\ &= \det\left(\mathbf{1}_E + \frac{i}{2\pi}F_A\right)\det\left(\mathbf{1}_F + \frac{i}{2\pi}F_B\right) \\ &= c(E)c(F).\end{aligned}$$

□

Example 16.11. Over $\mathbb{CP}^n$, we have:

$$c(\mathscr{O}(1) \oplus \mathscr{O}(-1)) = (1+x)(1-x) = 1 - x^2 = \begin{cases} 1, & \text{for } n = 1 \\ 1 - x^2, & \text{for } n \geq 2. \end{cases}$$

For $n = 1$, this must be true: we have the exact sequence (3.6) above, which splits by Corollary 3.4, so the bundle is trivial. On the other hand, since $x^2 \neq 0 \in H^4(\mathbb{CP}^n)$ for $n \geq 2$, this bundle is *not* trivial (even topologically).

Another result worth mentioning here is the following:

Theorem 16.12. *The integral cohomology ring of the complex infinite Grassmannian is a polynomial algebra freely generated by the Chern classes of the universal bundle:*

$$H^*(\mathrm{G}_r, \mathbb{Z}) = H^*(B\mathrm{GL}(n,\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}[\sigma_1(E_r), \dots, \sigma_r(E_r)].$$

In particular, the Chern classes give all characteristic classes of $\mathrm{GL}(n,\mathbb{C})$*-bundles.*

Proof. See Milnor and Stasheff, Theorem 14.5. The fact that the generators are algebraically independent follows from the rank-one case by an argument based on the Whitney product formula. To see that these generate the whole ring, one observes that G_r has exactly as many Schubert cells in each dimension as there are polynomials of the same degree in these generators. □

16.4. **A word about other structure groups.**

16.4.1. *The orthogonal group.* For $G = \mathrm{O}(r)$, we can define the **Pontryagin classes**, $p_k(E) \in H^{4k}(M)$, by

$$\left[\det\left(\mathbf{1} + \frac{1}{2\pi} F_A\right)\right] = \sum_{k=0}^{\lfloor r/4 \rfloor} p_k(E).$$

Note that for the above expression to make sense, we should have $[\sigma_{2k+1}(F_A)] = 0$ for each k. This is in fact the case: for instance, for $k = 1$, since F_A is a skew-symmetric real matrix, we have $\mathrm{Tr}_{\mathbb{R}} F_A = 0$. To see the vanishing in general, note that the Pontryagin classes can be given in terms of Chern classes by the formula

$$p_{k/2}(E) = (-i)^k c_k(E \otimes_{\mathbb{R}} \mathbb{C}),$$

which follows directly from the definition. Since $E \otimes_{\mathbb{R}} \mathbb{C} \cong \overline{E \otimes_{\mathbb{R}} \mathbb{C}}$, it follows from the definition of Chern classes that $c_{2k+1}(E \otimes_{\mathbb{R}} \mathbb{C}) = 0$ (exercise).

16.4.2. *The special orthogonal group.* Let $G = \mathrm{SO}(r)$, with r even. Then $\mathfrak{g}$ is again the subspace of skew-symmetric matrices. Up to conjugation, an element of $\mathfrak{g}$ will be block diagonal with 2×2 blocks of the form

$$\begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}.$$

The **Pfaffian**

$$\mathrm{Pf}(X) = \prod_i \lambda_i$$

is a degree $r/2$ polynomial which clearly satisfies

$$\mathrm{Pf}(X)^2 = \det X. \tag{16.2}$$

Since $\mathrm{SO}(r)$ acts on the subspace of block-diagonal matrices of the above form by permuting the blocks while changing the signs of an even number of them, the product is well-defined and Ad-invariant. For a given r, an expression for $\mathrm{Pf}(X)$ in terms of the matrix entries $X^\alpha{}_\beta$ can be found directly from (16.2).

The **Euler class** is defined by

$$e(E) = [\mathrm{Pf}(F_A)] \in H^r(M).$$

Since $\mathrm{U}(r/2) \subset \mathrm{SO}(r)$, the Euler class of a complex bundle is well-defined, and equal to none other than the top Chern class:

$$e(E) = c_r(E). \tag{16.3}$$

In the topological approach to characteristic classes (see Milnor-Stasheff), one first defines the Euler class of an oriented bundle using the Thom isomorphism, and later defines the

Chern classes by downwards induction from (16.3). The Euler class/top Chern class also has a simple interpretation in terms of the vanishing locus of a transverse section: see Griffiths and Harris.

The reason for the terminology is the following:

Theorem 16.13 (Chern-Gauss-Bonnet). *For a compact, smooth, orientable manifold, M, we have*

$$\int_M e(TM) = \chi(M).$$

Proof. See Milnor and Stasheff, Corollary 11.2 and Appendix C, p. 331. □

Note that if M is an almost-complex manifold, then the Euler class of the tangent bundle is nothing but $c_n(M) = e(TM)$, so this generalizes classical Gauss-Bonnet from last class. Also note that the result is only interesting if $\dim M$ is even; otherwise $\chi(M) = 0$, by Poincaré duality, and $\int_M e(TM) = 0$, either by definition or by Milnor and Stasheff, Property 9.4.

16.5. **Moral.** The above theorem says that the Euler characteristic can be determined by integrating certain universal expressions in the Riemann curvature tensor. On the other hand, the topology of the manifold imposes restrictions on the curvature of any possible metric. This interplay between curvature and topology is a central theme in differential geometry (including gauge theory).

For us, the main takeway is something more mundane:

Corollary of Chern-Weil theory. *If any Chern class of a complex vector bundle is nonzero, then it cannot admit a flat connection.*

Similarly, if any Pontryagin class of a real bundle is nonzero, then it cannot admit a flat connection.

If the Euler class of an orientable real bundle with metric is nonzero, then it cannot admit a metric-compatible flat connection.[20]

For this reason, we need to quantify the total amount of curvature that connections on a given bundle can carry; this is the role of the *Yang-Mills functional.*

16.6. **Exercises.**

1. Recall from Exercise 6.3.4. that
$$\mathscr{O}(2) \oplus \mathscr{O} \cong_{\mathbb{C}^\infty} \mathscr{O}(1) \oplus \mathscr{O}(1)$$
over $\mathbb{CP}^1$. Show that this fails over $\mathbb{CP}^n$ for $n \geq 2$.
2. Show that $c_{2k+1}(E \otimes_{\mathbb{R}} \mathbb{C}) = 0$, for each k.
3. For a 4×4 skew-symmetric real matrix, X, find a polynomial formula for $\mathrm{Pf}(X)$ in terms of its matrix coefficients.
4. Suppose that E is a complex vector bundle, and let $E_{\mathbb{R}}$ be the underlying real bundle. Show that $E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong E \oplus \bar{E}$. Use this to determine a formula for the Pontryagin classes of $E_{\mathbb{R}}$ in terms of the Chern classes of E. (See Milnor-Stasheff, Corollary 15.5, for such a formula.)

[20] Interestingly, this result fails when one removes "metric-compatible" from the statement: see the end of Appendix C in Milnor and Stasheff.

5. Show that the Euler class of a complex bundle is equal to the top Chern class. (Hint: this follows from the identity $\det_{\mathbb{R}} X = |\det_{\mathbb{C}} X|^2$, where one views $X \in \mathfrak{gl}(n, \mathbb{C}) \subset \mathfrak{gl}(2n, \mathbb{R})$. See my complex manifolds notes, Theorem 3.1.5, for a careful proof.)

Part III. The Yang-Mills functional

17. Definition and first properties (3/24-29)

17.1. **The definition.** Let g be a metric on TM and h a metric on $E \to M$, a real vector bundle with structure group $O(n)$ (as defined by h). Suppose that $\{\varphi^i\}$ is an orthonormal frame for T^*M. By convention, we let g induce the metric on $\Omega^k(M)$ in which

$$\{\varphi^{i_1} \wedge \cdots \wedge \varphi^{i_k}\}_{i_1 < \cdots < i_k}$$

is an orthonormal frame. Given a metric-compatible connection, A, the pointwise norm of the curvature tensor is

$$|F_A|^2 = \left| \sum_{\substack{i<j \\ \alpha,\beta}} F_{ij}{}^\alpha{}_\beta dx^i \wedge dx^j \otimes e_\alpha \otimes e^\beta \right|^2 = \sum_{\substack{i<j,k<\ell \\ \alpha,\beta,\gamma,\delta}} F_{ij}{}^\alpha{}_\beta F_{k\ell}{}^\delta{}_\gamma g^{ik} g^{j\ell} h_{\alpha\delta} h^{\beta\gamma},$$

where g^{ij} (resp. $h^{\alpha\beta}$) is the inverse matrix of g_{ij} (resp. $h_{\alpha\beta}$).

Definition 17.1. The **Yang-Mills functional** is given by

$$\mathcal{YM}(A) = \frac{1}{2} \int_M |F_A|^2 dV_g.$$

This expression is manifestly invariant under orthogonal gauge transformations. Consequently, it descends to a function on the moduli space of (metric-compatible) connections modulo gauge:

$$\mathcal{YM} : \mathscr{B}_E = \mathscr{A}_E / \mathscr{G}_E \to \mathbb{R}_{\geq 0}.$$

We had previously stated that classical gauge theory is the study of the space of connections modulo gauge. This wasn't quite precise: it is the study of the Yang-Mills functional on the space of metric-compatible connections modulo gauge.

17.2. **Yang-Mills connections.** Recall that if $E \to M$ is a real vector bundle, then

$$\{\text{flat connections on } E\}/\text{gauge} \cong \{\text{reps of } \pi_1(M)\}/\text{conjugacy}.$$

This is a nice, finite-dimensional variety inside the giant, mysterious infinite-dimensional space $\mathscr{B}_E$. So it would be very convenient if we could just restrict our attention to flat connections. However, we saw last time that any vector bundle with $c_i(E) \neq 0$, for some i, cannot carry a flat connection.

We can nonetheless look for connections with the "least amount" of curvature, *i.e.*, minimizers for the Yang-Mills functional $\mathcal{E} : \mathscr{B} \to \mathbb{R}_{\geq 0}$. This will be our goal.

The first step is to compute the first variation of $\mathcal{YM}$. Let

$$A_t = A + ta, \quad a \in \Omega^1(\mathfrak{g}_E).$$

If M is compact (so integrals are finite and integration by parts works) then

$$\begin{aligned} \frac{d}{dt} \mathcal{YM}(A_t)\Big|_{t=0} &= \frac{1}{2} \int_M \frac{d}{dt} \langle F_{A_t}, F_{A_t} \rangle \Big|_{t=0} dV \\ &= \int_M \left\langle \frac{d}{dt} F_{A_t}, F_{A_t} \right\rangle \Big|_{t=0} dV \end{aligned}$$

$$= \int_M \left\langle D_{A_t} \frac{d}{dt} A_t, F_{A_t} \right\rangle dV$$
$$= \int_M \langle D_A a, F_A \rangle \, dV$$
$$= \int_M \langle a, D_A^* F_A \rangle \, dV.$$

Here,

$$D_A^* : \Omega^2(\mathfrak{g}_E) \to \Omega^1(\mathfrak{g}_E)$$

is the **formal adjoint** of D_A. We will see below in Definition/Lemma 17.4 that this is a concrete differential operator.

Definition 17.2. A is called a **Yang-Mills connection**, *i.e.*, a critical point of the Yang-Mills functional, if and only if

$$D_A^* F_A = 0.$$

The remainder of this section is devoted to writing down this PDE explicitly.

17.2.1. *The formal adjoint.* Let φ^i be an orthonormal frame. The **Hodge $\star$-operator**

$$\star : \Omega^k \to \Omega^{n-k}$$

is the unique linear operator such that

$$\star(\varphi^{i_1} \wedge \cdots \wedge \varphi^{i_k}) = \pm \varphi^{i_{k+1}} \wedge \cdots \varphi^{i_n},$$

where the $\pm$ and $\{i_{k+1}, \ldots, i_n\}$ are determined by the requirement

$$\pm \varphi^{i_1} \wedge \cdots \wedge \varphi^{i_k} \wedge \varphi^{i_{k+1}} \wedge \cdots \varphi^{i_n} = d\mathrm{Vol}.$$

This operator is characterized by the following property: if α and β are real-valued k-forms, then we have

$$\langle \alpha, \beta \rangle \, d\mathrm{Vol} = \alpha \wedge \star\beta.$$

We define

$$\star : \Omega^k(E) \to \Omega^{n-k}(E)$$

by extending linearly; in other words, we let $\star$ act on the form component without touching the bundle component.

Lemma 17.3. *$\star^2 = (-1)^{k(n-k)}$ on k-forms.*

Proof. Exercise. $\square$

Consider the special case of $\Omega^k(\mathfrak{g}_E)$. Here, as always, we are assuming that the structure group is $\mathrm{O}(n)$ or a Lie subgroup, so elements of $\mathfrak{g}$ are skew-symmetric matrices. Let $\{e_\alpha\}$ be an orthonormal frame for E and $u, v \in \Gamma(\mathfrak{g}_E)$. The local components are given by

$$u = u^\alpha{}_\beta e_\alpha \otimes e^\beta, \quad v = v^\alpha{}_\beta e_\alpha \otimes e^\beta,$$

where we go back to using the Einstein convention. By skew-symmetry, we have

$$\langle u, v \rangle = u^\alpha{}_\beta v^\alpha{}_\beta = -u^\alpha{}_\beta v^\beta{}_\alpha = -\operatorname{Tr}(u \cdot v).$$

A k-form $\omega \in \Omega^k(\mathrm{End} E)$ has components

$$\omega = \omega^\alpha{}_\beta e_\alpha \otimes e^\beta = \frac{1}{k!}\omega_{i_1\cdots i_k}{}^\alpha{}_\beta\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \otimes e_\alpha \otimes e^\beta.$$

Recall that the wedge product between endomorphism-valued forms was defined in (14.1) by

$$\omega \wedge \eta = (\omega^\alpha{}_\gamma \wedge \eta^\gamma{}_\beta) e_\alpha \otimes e^\beta.$$

For $\omega, \eta \in \Omega^k(\mathfrak{g}_E) \subset \Omega^k(\mathrm{End} E)$, we have

$$\begin{aligned} \langle \omega, \eta\rangle\, d\mathrm{Vol} = \langle \omega^\alpha{}_\beta, \eta^\alpha{}_\beta\rangle\, d\mathrm{Vol} &= \omega^\alpha{}_\beta \wedge *\eta^\alpha{}_\beta \\ &= -\omega^\alpha{}_\beta \wedge *\eta^\beta{}_\alpha \\ &= -\mathrm{Tr}\,\omega \wedge *\eta. \end{aligned} \tag{17.1}$$

Definition/Lemma 17.4. *The formal adjoint $D_A^* : \Omega^k(\mathfrak{g}_E) \to \Omega^{k-1}(\mathfrak{g}_E)$ is given by*

$$D_A^* = (-1)^{1+n(k+1)} * D_A *$$

and satisfies

$$\int_M \langle D_A \mu, \omega\rangle\, dV = \int_M \langle \mu, D_A^* \omega\rangle\, dV,$$

for compactly supported forms $\mu \in \Omega^{k-1}(\mathfrak{g}_E)$ and $\omega \in \Omega^k(\mathfrak{g}_E)$.

Proof. We have

$$d\,\mathrm{Tr}(\mu \wedge *\omega) = \mathrm{Tr}(D_A\mu \wedge *\omega) + (-1)^{k-1}\,\mathrm{Tr}(\mu \wedge D_A * \omega).$$

Integrating both sides gives

$$0 = \int \mathrm{Tr}(D_A\mu \wedge *\omega) + (-1)^{k-1}\,\mathrm{Tr}(\mu \wedge D_A * \omega).$$

Moving over the left-hand term, we have

$$\begin{aligned} -\int \mathrm{Tr}(D_A\mu, \omega) = \int \langle D_A\mu, \omega\rangle\, dV &= (-1)^{k-1}\int \mathrm{Tr}(\mu \wedge D_A * \omega) \\ &= (-1)^k \int \left\langle \mu, *^{-1} D_A * \omega\right\rangle\, dV \\ &= (-1)^{k+(n-k+1)(k-1)} \int \langle \mu, * D_A * \omega\rangle\, dV, \end{aligned}$$

by Lemma 17.3, since $D_A * \omega$ is an $n-k+1$-form. Note that

$$k + (n-k+1)(k+1) \equiv 1 + (n-k+2)(k+1) \equiv 1 + n(k+1) + k(k+1) \equiv 1 + n(k+1) \mod 2,$$

so this gives the desired formula. $\square$

The Yang-Mills equation says that $D_A^* F_A = 0$. By the Definition/Lemma and Exercise 17.3, this means

$$0 = \pm * D_A * F_A.$$

But $\pm *$ is an isomorphism, so this is equivalent to

$$D_A * F_A = 0.$$

The Bianchi identity tells us $D_A F_A = 0$. Put together, the two equations

$$D_A * F_A = 0, \qquad D_A F_A = 0$$

just state that F_A is a **harmonic 2-form** (with respect to A and the chosen metrics on the bundle and base manifold). So the equation is very natural from a geometric/analytic perspective, once we have the right framework within which to state it.

17.2.2. *Another expression for the formal adjoint.* We'll rewrite the Yang-Mills equation in a different and simpler-looking form.

Recall that the covariant exterior derivative was originally defined by the composition

$$D_A : \Omega^k(\mathfrak{g}_E) \xrightarrow{\nabla_A} T^*M \otimes \Omega^k(\mathfrak{g}_E) \xrightarrow{\wedge} \Omega^{k+1}(\mathfrak{g}_E).$$

If we choose a section of $\wedge : T^*M \otimes \Omega^k(\mathfrak{g}_E) \to \Omega^{k+1}(\mathfrak{g}_E)$, for instance by identifying both with subspaces of the full tensor product $T^*M^{\otimes k}$, then the adjoint should just be given by the restriction

$$D_A^* = \nabla_A^* \Big|_{\Omega^k(\mathfrak{g}_E) \subset (T^*M)^{\otimes k}}.$$

This identification can be a bit confusing when metrics are involved, so we shall instead make the following explicit calculation.

Recall our convention that we always take the tensor components of a k-form to be alternating; so we have

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1,\ldots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \frac{1}{k!} \omega_{i_1\ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where the indices in the rightmost term run over all values of $i_1, \ldots, i_k$ according to the usual Einstein summation convention. Choose geodesic coordinates at a point $p \in M$, so that $g_{ij}(p) = \delta_{ij}$. We have the following expression for the inner product between two k-forms at p:

$$\langle \omega, \eta \rangle = \sum_{i_1 < \cdots < i_k} \langle \omega_{i_1 \cdots i_k}, \eta_{i_1 \cdots i_k} \rangle = \frac{1}{k!} \langle \omega_{i_1 \cdots i_k}, \eta_{i_1 \cdots i_k} \rangle, \tag{17.2}$$

where we abuse the Einstein summation convention on the RHS. Given a $(k-1)$-form

$$\mu = \frac{1}{(k-1)!} \mu_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

we have

$$\begin{aligned} D_A \mu &= \frac{1}{(k-1)!} \nabla_{i_1} \mu_{i_2 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &= \frac{1}{k!} \left(\nabla_{i_1} \mu_{i_2 \cdots i_k} - \nabla_{i_2} \mu_{i_1 i_3 \cdots i_k} - \cdots - \nabla_{i_k} \mu_{i_2 \cdots i_1} \right) dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \end{aligned} \tag{17.3}$$

where we have used the fact that $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ is alternating in $i_1, \ldots, i_k$. The coefficients are now alternating, so these are the genuine components of the k-form $D_A \mu$. By (17.2), we have

$$\begin{aligned} \langle D_A \mu, \omega \rangle &= \frac{1}{k!} \langle \nabla_{i_1} \mu_{i_2 \cdots i_k} - \nabla_{i_2} \mu_{i_1 i_3 \cdots i_k} - \cdots - \nabla_{i_k} \mu_{i_2 \cdots i_1}, \omega_{i_1 \cdots i_k} \rangle \\ &= \frac{k}{k!} \langle \nabla_{i_1} \mu_{i_2 \cdots i_n}, \omega_{i_1 \cdots i_k} \rangle = \frac{1}{(k-1)!} \langle \nabla_{i_1} \mu_{i_2 \cdots i_n}, \omega_{i_1 \cdots i_k} \rangle, \end{aligned}$$

since $\omega_{i_1\cdots i_k}$ is alternating. Reverse-engineeering the Leibniz rule, we have

$$\begin{aligned}\langle D_A\mu,\omega\rangle &= \frac{1}{(k-1)!}\left(\nabla_{i_1}\langle\mu_{i_2\cdots i_k},\omega_{i_1 i_2\cdots i_k}\rangle - \langle\mu_{i_2\cdots i_k},\nabla_{i_1}\omega_{i_1\cdots i_k}\rangle\right)\\ &= \operatorname{div}(\text{something}) - \frac{1}{(k-1)!}\langle\mu_{i_2\cdots i_k},\nabla_{i_1}\omega_{i_1\cdots i_k}\rangle .\end{aligned}$$

Note that the divergence term vanishes after integration, and by (17.2), the second term is just the inner product between $(k-1)$-forms. So we obtain another expression for the formal adjoint:

$$(D_A^*\omega)_{i_2\cdots i_k} = -\nabla_i\omega_{ii_2\cdots i_k} = -g^{ij}\nabla_i\omega_{ji_2\cdots i_k}. \tag{17.4}$$

Using this expression, the Yang-Mills equation is just:

$$\nabla^i F_{ik} = 0. \tag{17.5}$$

While this expression looks simpler than that of Definition 17.4, we are using the Levi-Civita connection to differentiate cotangent-bundle indices. Also note that in this calculation, we have implicitly applied "Ricci's Lemma" several times; this says that $\nabla_i g_{jk} = 0$ for the Levi-Civita connection (exercise), so one can always pass the metric through a covariant derivative.

Remark 17.5 (Raising and lowering indices)**.** Here is a remark for those of you who never took a first course in general relativity (or a second course in differential geometry). As we know, a metric g on E induces an isomorphism

$$\begin{aligned} E &\to E^* \\ s &\mapsto \langle -, s\rangle_g .\end{aligned}$$

The coordinate expression for the image is obtained by "lowering indices," as follows. In a local frame, write

$$g_{\alpha\beta} = \langle e_\alpha, e_\beta\rangle$$

for the components of the metric tensor (a section of $E^*\otimes E^*$), and

$$g^{\alpha\beta}$$

for the inverse matrix of $g_{\alpha\beta}$, which gives a section of $E\otimes E$. Given a section $s = s^\alpha e_\alpha$ of E, we can "lower the index" of s by setting

$$s_\alpha := g_{\alpha\beta}s^\beta.$$

This gives us a section $s_\alpha e^\alpha$ of E^*, where $\{e^\alpha\}$ is the dual frame of $\{e_\alpha\}$. This is the image of s under the above isomorphism.

We can also "raise the index" back as follows:

$$g^{\alpha\beta}s_\beta = g^{\alpha\beta}(g_{\beta\gamma}s^\gamma) = \delta^\alpha{}_\gamma s^\gamma = s^\alpha.$$

Since raising the index on s_α gives us back the original components s^α, the notation is well-defined.

17.3. **Exercises.**

1. Prove Lemma 17.3.
2. Show that the coefficients in the last line of (17.3) are indeed alternating.
3. Prove "Ricci's Lemma," which states that $\nabla_i g_{jk} = 0$ for the Levi-Civita connection. (Hint: this just amounts to the statement that ∇ is metric-compatible.)
4. Use "Ricci's Lemma" to justify the calculations leading to (17.4), which were based on assuming $g_{ij} = \delta_{ij}$.
5. Write out the full Yang-Mills equation as a PDE in local components / coordinates, involving g_{ij}, $A^\alpha_{i\beta}$, and (if necessary) Γ^k_{ij}. Check that the expression obtained using Definition 17.4 agrees with that of (17.5).

18. Maxwell's equations and the magnetic monopole (3/29-31)

Example 18.1. Let $G = \mathrm{U}(1)$, so $\mathfrak{g} = i\mathbb{R}$. A connection on $M = \mathbb{R}^3$ may be written

$$A = i(A_1 dx^1 + A_2 dx^2 + A_3 dx^3).$$

The 3-vector $\vec{A} = (A_1, A_2, A_3)$ is called the **vector potential** by physicists.

The curvature of A is

$$F = dA = i(\partial_1 A_2 - \partial_2 A_1) dx^1 \wedge dx^2 + i(\partial_1 A_3 - \partial_3 A_1) dx^1 \wedge dx^3 + i(\partial_2 A_3 - \partial_3 A_2) dx^2 \wedge dx^3.$$

Define $\vec{B} = (B_1, B_2, B_3)$ by

$$(F_{ij}) = i \begin{pmatrix} 0 & B_2 & -B_3 \\ -B_2 & 0 & B_1 \\ B_3 & -B_1 & 0 \end{pmatrix},$$

or in classical notation,

$$\vec{B} = \nabla \times \vec{A}.$$

This explains why the letter F is used for the curvature of a connection: it stands for "field-strength tensor."

We have

$$*F = i\left(B_1 dx^1 + B_2 dx^2 + B_3 dx^3\right).$$

The Yang-Mills equation reads:

$$d * F = 0 \iff \vec{\nabla} \times \vec{B} = 0.$$

The Bianchi identity reads:

$$dF = 0 \iff \vec{\nabla} \cdot \vec{B} = 0.$$

Together, these are just Maxwell's equations with electric field $\vec{E} \equiv 0$.

Note that "gauge invariance" is just the familiar statement from electromagnetism that $\vec{B}$ is unchanged by sending A to $A + du$.

Example 18.2. Consider a connection over $\mathbb{R}^{3,1}$ given by

$$A = i\left(\rho dt + A_1 dx^1 + A_2 dx^2 + A_3 dx^3\right).$$

Here, ρ is the so-called "electric potential," and $\vec{A}$ is the vector potential as above. Define $\vec{E}$ and $\vec{B}$ by

$$(F_{ij}) = i\begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_2 & -B_3 \\ -E_2 & -B_2 & 0 & B_1 \\ -E_3 & B_3 & -B_1 & 0 \end{pmatrix}.$$

In other words, $\vec{E} = \nabla\rho$ and $\vec{B} = \nabla\times\vec{A}$. Calculating out the Yang-Mills equations and Bianchi identity in Minkowski signature, one finds the four Maxwell equations, coming in pairs:

$$\begin{aligned} \nabla\times\vec{B} &= \frac{\partial\vec{E}}{\partial t} \\ \nabla\cdot\vec{E} &= 0 \end{aligned} \iff d * F = 0$$

$$\begin{aligned} \nabla\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} \\ \nabla\cdot\vec{B} &= 0 \end{aligned} \iff dF = 0.$$

Remark 18.3 (The Dirac equation)**.** Let A be a connection and ψ be a spinor field on $\mathbb{R}^{3,1}$, which is given by a complex 4-vector. According to Dirac, ψ should describe the wavefunction of an electron in the electromagnetic field A, subject to the **Dirac equation**

$$i \not{\nabla}_A \psi = m\psi.$$

Here m is the mass and

$$i \not{\nabla}_A \psi = i\gamma^\mu \cdot (\partial_\mu + A_\mu)\psi,$$

where γ^μ are the four 4×4 "Dirac matrices." Notice that A appears here as part of the equation, unlike in Maxwell's equations. Dirac was the first to posit that A might have a physical meaning, rather than just being a convenient mathematical device, as was thought classically.

Example 18.4 (Dirac's magnetic monopole)**.** Over $\mathbb{R}^3 \smallsetminus \{0\}$, let

$$\vec{E} \equiv 0, \quad \vec{B} = \frac{q\vec{r}}{r^3}.$$

This represents a magnetic monopole of charge q; notice that the field strength goes as $1/r^2$, exactly as for an electric monopole. Viewing B as a 1-form, we have

$$-i * F = B = q\frac{x^i dx^i}{r^3} = -qd\left(\frac{1}{r}\right),$$

which is closed. We also have

$$-idF = d * B = qd * d\left(\frac{1}{r}\right) = -\Delta\left(\frac{1}{r}\right) = 0,$$

so this is a solution of Maxwell's equations.

Is F the curvature of a connection A on $\mathbb{R}^3 \smallsetminus 0$? Dirac was motivated to ask this question (in his own language) just based on trying to write down his equation in the presence of a magnetic monopole, in a consistent way. This was the beginning of gauge theory, in 1931.

Let θ, ϕ be spherical coordinates, where ϕ measures rotation around the z-axis and θ measures angle from the z-axis. We have

$$F = i * B = iq \sin\theta \, d\theta \wedge d\phi.$$

Set

$$U_0 = M \smallsetminus \{(0,0,z) : z \le 0\}$$
$$U_1 = M \smallsetminus \{(0,0,z) : z \ge 0\}.$$

On U_0, set

$$A_0 = q(1 - \cos\theta) d\phi,$$

which extends smoothly across the positive z-axis. On U_1, set

$$A_1 = -q(1 + \cos\theta) d\phi,$$

which extends across the negative z-axis. It's clear that $dA_i = F$ on U_i.

To have this define a connection, we need

$$A_0 - A_1 = \sigma^{-1} d\sigma = d \log \sigma$$

for some $\mathrm{U}(1)$-valued function σ on $U_0 \cap U_1$. We have

$$A_0 - A_1 = 2iq \, d\phi = d(2iq\phi),$$

so we must have

$$\log \sigma = 2iq\phi + i(\text{constant}) \Rightarrow \sigma = ce^{2iq\phi}.$$

This is a well-defined function on $U_0 \cap U_1$ if and only if

$$2q \in \mathbb{Z}.$$

So this is a necessary and sufficient condition for the monopole with charge q to be the curvature of a connection on a $\mathrm{U}(1)$-bundle.

Dirac reached the same conclusion by considering his equation for two wavefunctions on U_0 and U_1, which needed to be related by a well-defined phase factor σ over $U_0 \cap U_1$. (Recall from quantum mechanics that two wavefunctions that differ only by a phase factor are supposed to represent the same physical state—this is how physicists think of gauge invariance.) He concluded that the charge of a magnetic monopole must be quantized, if such a thing exists. Really, he had just discovered the first Chern class.

Example 18.5. Define a connection on $\mathscr{O}(k) \to \mathbb{CP}^1 \cong S^2$ by

$$A^0 = \frac{ik \operatorname{Im}(z \, d\bar{z})}{1 + |z|^2} \text{ on } U_0$$

and

$$A^1 = -\frac{ik \operatorname{Im}(w d\bar{w})}{1 + |w|^2} \text{ on } U_1.$$

This is just a $\mathfrak{u}(1)$ version of the connection we considered during the proof of Lemma 14.7 above. Carrying over the proof there (exercise), we can calculate the curvature

$$F_A = i\frac{k\,dz \wedge d\bar{z}}{(1+|z|^2)^2} = -ikdV_{S^2},$$

where dV_{S^2} is the volume form of S^2 in stereographic coordinates (a.k.a. the Fubini-Study form). Hence, $*_{S^2} F_A = -ik$, and $d *_{S^2} F_A = 0$, so this is a solution of the Yang-Mills equation on S^2. In fact, the pullback of A by the radial projection $\mathbb{R}^3 \smallsetminus \{0\} \to S^2$ is just the Dirac monopole, with $k = \pm 2q$.

18.1. **Exercises.**

1. Do the calculations to show that the Yang-Mills equations for a U(1)-bundle over $\mathbb{R}^{3,1}$ are equivalent to the full Maxwell's equations. (Note: you have to be slightly careful with the Hodge star operator in Minkowski signature.)
2. Calculate the curvature of the connection in Example 18.5.
3. Show that the Dirac monopole is the pullback of the connection of Example 18.5 by the projection $\mathbb{R}^3 \smallsetminus \{0\} \to S^2$.

19. Yang-Mills connections and minimizers in 2D (3/31)

Let $M = \Sigma$ be a compact, oriented, connected, 2-dimensional Riemannian manifold.

Definition 19.1. For a complex bundle $E \to \Sigma$, we define

$$\deg(E) = \int_\Sigma c_1(E) \in \mathbb{Z}.$$

Given $E \to \Sigma$, with connection A, we have $F_A \in \Omega^2(\mathfrak{g}_E)$, so $*F_A \in \Omega^0(\mathfrak{g}_E) = \Gamma(\mathfrak{g}_E)$. Since $*F_A$ is a zero-form, the Yang-Mills equation $D_A * F_A = 0$ in fact implies

$$\nabla_A * F_A = 0.$$

Hence, $*F_A$ is covariantly constant, as is $F_A = *F_A\, dV$. We conclude that the curvature of any Yang-Mills connection over a Riemann surface must be covariantly constant.

Given $x \in \Sigma$, let $V = E_x$. Since $*F_A$ is skew-adjoint, it has a complete set of eigenvectors with eigenvalues $i\lambda_k$, where $\lambda_k \in \mathbb{R}$, with multiplicity r_k. Let V_k be the $i\lambda_k$-eigenspace. We can decompose $V = \bigoplus_k V_k$, and $*F_A = \bigoplus_k i\lambda_k \mathbf{1}_{V_k}$. But $*F$ is covariantly constant, so the eigenvalues are also covariantly constant and the eigenspaces are preserved by parallel transportation. Consequently, the V_k's at each point form a subbundle $E_k \subset E$, and we have

$$E = \bigoplus_k E_k,$$

with

$$F = \bigoplus_k i\lambda_k \mathbf{1}_{E_k} \otimes dV.$$

Also note that

$$\deg(E_k) = \frac{i}{2\pi}\int \operatorname{Tr} F|_{E_k} = -\frac{\lambda_k r_k}{2\pi}\operatorname{Vol}(\Sigma),$$

so the values of λ_k are determined by the degree and rank of E_k. This proves the first statement of the following theorem.

Theorem 19.2. *Assume* $\mathrm{Vol}(\Sigma) = 1$. *Given* $E \to \Sigma$, *any Yang-Mills connection* A *splits* E *into* $\bigoplus_k E_k$, *where*

$$F_A = \bigoplus_k \frac{2\pi \deg(E_k)}{i\,\mathrm{rk}(E_k)} \mathbf{1}_{E_k} \otimes dV.$$

Moreover, A *minimizes* $\mathcal{YM}$ *if and only if* $k = 1$ *and*

$$F_A = \frac{2\pi \deg(E)}{i\,\mathrm{rk}(E)} \mathbf{1}_E \otimes dV.$$

Remark 19.3. The quantity

$$\frac{\deg(E)}{\mathrm{rk}(E)} =: \mu(E)$$

is called the **slope** of E.

A connection A whose curvature solves (19.2) is called **projectively flat**.

Proof of "moreover". Recall that $*F$ is an $r \times r$ matrix, and $|F|^2 = |*F|^2$. For any matrix X,

$$|X|^2 = X^\alpha{}_\beta \bar{X}^\alpha{}_\beta \geq \frac{1}{n} (\mathrm{Tr}\, X)^2.$$

This is from Cauchy-Schwarz:

$$\mathrm{Tr}\, X = X^\alpha{}_\alpha = X^\alpha{}_\beta \delta^\alpha{}_\beta = \langle X, \mathbf{1} \rangle \leq |X| \sqrt{n}.$$

Hence

$$\frac{(\mathrm{Tr}\, X)^2}{n} \leq |X|^2.$$

Also according to Cauchy-Schwarz, equality holds iff X is a constant times $\mathbf{1}$. The value of this constant is fixed by the first part of the Theorem. □

As a corollary of the proof, we have:

Corollary 19.4. *For any connection* A *on* $E \to \Sigma$, *where* $\mathrm{Vol}\Sigma = 1$, *we have*

$$\mathcal{YM}(A) \geq 4\pi^2 \frac{\deg(E)^2}{\mathrm{rk}(E)},$$

with equality if and only if A *is projectively flat.*

Example 19.5. Suppose $\mathrm{rk}(E) = 1$. Pick A_0 any connection on $E \to \Sigma$. We have

$$F_{A_0 + a} = F_{A_0} + D_{A_0} a + a \wedge a = F_{A_0} + da,$$

because $a \wedge a = 0$ and the endomorphism bundle of a rank 1 bundle is trivial.

We shall choose a to be of the form

$$a = *df$$

for $f \in C^\infty(\Sigma)$. Then

$$*(F_{A_0 + a}) = *F_{A_0} + *d * df = *F_{A_0} - \Delta f.$$

We claim that there exists f such that

$$\Delta f = *F_{A_0} + 2\pi i \deg E. \tag{19.1}$$

This is because

$$\int *F_{A_0} dV = \int F_{A_0} = -2\pi i \deg E \implies \int \left(*F_{A_0} + 2\pi i \deg E\right) dV = 0.$$

Hence, the RHS of (19.1) has average zero. Since Σ is a compact manifold, this is necessary and sufficient to solve (19.1).

Taking f to be the solution of (19.1), we get

$$F_{A_0+a} = F_{A_0} + *\Delta f = F_{A_0} - \left(F_{A_0} + 2\pi i \deg E\right) = \frac{2\pi}{i} \deg(E).$$

Hence, minimizers exist on line bundles over Riemann surfaces.

Example 19.6. Recall that all higher-rank complex bundles over Σ split topologically as

$$E = L \oplus \underline{\mathbb{C}}^{r-1}.$$

Take A on L as in the previous example. Then $A \oplus$ (product connection) is Yang-Mills, but not a minimizer.

Example 19.7. Since $\mathscr{O}(1) \oplus \mathscr{O}(-1) \cong_{C^\infty} \underline{\mathbb{C}}^2 \to \mathbb{CP}^1$, this bundle carries both a (trivial) minimizer, and a non-minimizing Yang-Mills connection given by the direct sum of two of the connections from Example 18.5 with opposite charges.

In general, a minimizer does exist for any bundle over a compact Riemann surface. This can be shown using Uhlenbeck's theorem (which also works in dimension 3), relying on heavy analysis, or directly using principal bundle formalism, as in Atiyah-Bott, §6. However, this is all much too trivial, since really we are just looking for a modified type of flat connection. The Yang-Mills equation on a Riemann surface is only interesting when you consider the interplay with holomorphic structures, as we'll do later.

19.1. **Exercises.**

1. * Suppose that A is a non-minimizing Yang-Mills connection over Σ, *i.e.*, a connection whose curvature is a direct sum of at least two factors with different slopes. Can you write down a path of connections starting from A that decreases the Yang-Mills functional?
2. * Read Atiyah-Bott, §6.

20. Instantons in 4D (4/5)

20.1. **Instantons.**

Suppose that $M = M^4$ is oriented with metric g. The Hodge star operator

$$\star : \Omega^2(M) \to \Omega^2(M)$$

now acts on 2-forms, with $*^2 = (-1)^{2(4-2)} = 1$. We further have

$$\langle \alpha, *\beta \rangle \, dV = \alpha \wedge \beta = \beta \wedge \alpha = \langle \beta, *\alpha \rangle \, dV = \langle *\alpha, \beta \rangle \, dV$$

so $*$ is self-adjoint. Since $*^2 = \mathbf{1}$, the eigenvalues of $*$ are ± 1, and we obtain an orthogonal decomposition

$$\Omega^2 = \Omega^{2+} \oplus \Omega^{2-},$$

where $\Omega^{2\pm}$ denotes the ±1 eigenspace of $*$.

Let $\mathbb{R}^4$ have coordinates $x^0, \ldots, x^3$. Observe that

$$*(dx^0 \wedge dx^1) = dx^2 \wedge dx^3.$$

We have the following explicit frames for Ω^{2+},

$$dx^0 \wedge dx^1 + dx^2 \wedge dx^3, \quad dx^0 \wedge dx^2 - dx^1 \wedge dx^3, \quad dx^0 \wedge dx^3 + dx^1 \wedge dx^2,$$

and for Ω^{2-},

$$dx^0 \wedge dx^1 - dx^2 \wedge dx^3, \quad dx^0 \wedge dx^2 + dx^1 \wedge dx^3, \quad dx^0 \wedge dx^3 - dx^1 \wedge dx^2.$$

Definition 20.1. A connection on $E \to M^4$ is called an **instanton** if $F_A \in \Omega^{2+}(\mathfrak{g}_E)$ or $F_A \in \Omega^{2-}(\mathfrak{g}_E)$. The former are called *self-dual* and the latter are called *anti-self-dual.*

Proposition 20.2. *Instantons are Yang-Mills.*

Proof. $D_A * F_A = \pm D_A F_A = 0$ by Bianchi identity. $\square$

We can decompose F_A into self-dual and anti-self-dual parts as follows:

$$F_A = F_A^+ + F_A^- = \frac{1}{2}(F_A + *F_A) + \frac{1}{2}(F_A - *F_A).$$

Notice that $F_A = F_A^-$ iff $F_A^+ = 0$ iff

$$F_A + *F_A = 0.$$

This "anti-self-duality equation" is a first-order PDE in A, whereas the general Yang-Mills equation is a 2nd-order PDE in A. So we say that the ASD equation is a "first-order reduction" of the Yang-Mills equation.

Recall (perhaps from Example 19.6) that not all Yang-Mills connections are minimizers of $\mathcal{YM}$. Nevertheless, we have:

Proposition 20.3. *Instantons are minimizers of $\mathcal{YM}$ on a compact 4-manifold. Specifically, for connections with structure group* $\mathrm{SU}(r)$*, we have*

$$\mathcal{YM}(A) \geq 4\pi^2 \int_M c_2(E), \tag{20.1}$$

with equality iff A is anti-self-dual.

Proof. Recall that for $\xi \in \mathfrak{su}(r) \subset \mathfrak{u}(r) \subset \mathfrak{so}(2r)$, we have

$$-\operatorname{Tr}\xi^2 = |\xi|^2.$$

Let $F_A = F_A^+ + F_A^-$. Then

$$\begin{aligned}\operatorname{Tr}(F_A \wedge F_A) &= \operatorname{Tr} F_A^+ \wedge F_A^+ + F_A^- \wedge F_A^- \\ &= -\langle F_A^+, *F_A^+\rangle\, dV - \langle F_A^-, *F_A^-\rangle\, dV \\ &= (-|F_A^+|^2 + |F_A^-|^2)\, dV.\end{aligned}$$

On the other hand, since F_A^+ and F_A^- are orthogonal, we have

$$|F_A|^2 = |F_A^+|^2 + |F_A^-|^2.$$

Adding $2|F_A^+|^2 dV$ to the first formula, we obtain the RHS of the second formula; so

$$|F_A|^2 dV = \mathrm{Tr}(F_A \wedge F_A) + 2|F_A^+|^2 dV.$$

Integrating yields

$$\mathcal{YM}(A) = \frac{1}{2}\int |F_A|^2 dV = \frac{1}{2}\int \mathrm{Tr}(F_A \wedge F_A) + \int |F_A^+|^2\, dV.$$

By Example 16.8, on an $\mathrm{SU}(r)$-bundle, the RHS is

$$4\pi^2 \langle c_2(E), M\rangle + \int |F_A^+|^2.$$

This implies (20.1), with equality iff $F_A^+ \equiv 0$. $\square$

20.2. **Extended example: the standard instanton.** Recall the following from Section §6.2. Hamilton's quaternions $\mathbb{H}$ are a 4-dimensional real algebra generated by q_0, q_i for $i = 1, 2, 3$, subject to the relations

$$\begin{aligned} q_0 &= \mathbf{1} \\ q_i^2 &= -\mathbf{1}, \quad i = 1,2,3 \\ q_1 q_2 q_3 &= \mathbf{1}. \end{aligned}$$

For $a = a^i q_i \in \mathbb{H}$, the norm is given by

$$|a|^2 = a\bar{a} = \bar{a}a = \sum |a_i|^2 \in \mathbb{R}.$$

The real and imaginary parts of a are

$$\mathrm{Re}(a) = \frac{a+\bar{a}}{2} \quad \text{and} \quad \mathrm{Im}(a) = \frac{a-\bar{a}}{2} \in \mathbb{R}\cdot\{q_1, q_2, q_3\} \cong \mathbb{R}^3 \subset \mathbb{H} \cong \mathbb{R}^4.$$

The unit sphere $S^3 \subset \mathbb{H}$ forms a group isomorphic to $\mathrm{SU}(2)$. (You can see the isomorphism by letting the unit sphere act on $\mathbb{H}$ by left-multiplication.) The exponential map identifies the imaginary quaternions $\mathrm{Im}\,\mathbb{H}$ with the Lie algebra $\mathfrak{su}(2)$.

We define an $\mathbb{H}$-valued function on $\mathbb{H} \cong \mathbb{R}^4$ by:

$$x = x^i q_i.$$

The following is an $\mathbb{H}$-valued 1-form on $\mathbb{R}^4$:

$$dx = q_i dx^i.$$

Consider the $\mathbb{H}$-valued 2-form

$$dx \wedge d\bar{x} = (q_0 dx^0 + q_1 dx^1 + \cdots) \wedge (q_0 dx^0 - q_1 dx^1 - \cdots).$$

Here we are combining wedge product with quaternion multiplication, as we always do for matrix-valued (or algebra-valued) forms. Expanding yields

$$\begin{aligned} &dx^0 \wedge dx^1(-2q_1) + dx^0 \wedge dx^2(-2q_2) + dx^0 \wedge dx^3(-2q_3) \\ &\qquad - [q_1, q_2]dx^1 \wedge dx^2 - [q_1, q_3]dx^1 \wedge dx^3 - [q_2, q_3]dx^2 \wedge dx^3 \\ &= -2\Big(q_1(dx^0 \wedge dx^1 - dx^2 \wedge dx^3) + q_2(dx^0 \wedge dx^2 + dx^1 \wedge dx^3) + q_3(dx^0 \wedge dx^3 - dx^1 \wedge dx^2)\Big). \end{aligned}$$

Comparing with the standard basis of anti-self-dual 2-forms above, we see that $dx \wedge d\bar{x} \in \Omega^{2-}(\operatorname{Im}\mathbb{H})$ is an $\operatorname{Im}\mathbb{H}$-valued anti-self-dual 2-form. Similarly, one can show that $d\bar{x} \wedge dx \in \Omega^{2+}(\operatorname{Im}\mathbb{H})$ is a self-dual 2-form (exercise).

Define a connection on $\mathbb{R}^4$ by

$$A(x) = \frac{\operatorname{Im}(x\, d\bar{x})}{1 + |x|^2}.$$

We have

$$dA = \operatorname{Im}\left[\frac{dx \wedge d\bar{x}}{1+|x|^2} + xd(1+|x|^2)^{-1} \wedge d\bar{x}\right].$$

Simplifying the second term yields

$$xd(1+|x|^2)^{-1} \wedge d\bar{x} = \frac{-x(d\bar{x}x + \bar{x}\, dx) \wedge d\bar{x}}{(1+|x|^2)^2} = \frac{-1}{(1+|x|^2)^2}\left[xd\bar{x} \wedge xd\bar{x} + |x|^2 dx \wedge d\bar{x}\right]. \tag{20.2}$$

Also

$$A \wedge A = \operatorname{Im}\left(\frac{xd\bar{x} \wedge xd\bar{x}}{(1+|x|^2)^2}\right),$$

since for $\alpha \in \Omega^1(\mathbb{H})$, $\operatorname{Im}(\alpha \wedge \alpha) = \operatorname{Im}(\alpha) \wedge \operatorname{Im}(\alpha)$ (exercise). Assembling the parts of F_A, we see that $A \wedge A$ cancels with the second term in (20.2). We obtain:

$$F_A = dA + A \wedge A = \frac{1}{(1+|x|^2)^2}\operatorname{Im}(dx \wedge d\bar{x}) = \frac{dx \wedge d\bar{x}}{(1+|x|^2)^2}.$$

(In the last equality we have removed the Im because $dx \wedge d\bar{x}$ is already purely imaginary, as we computed above.) Hence, A is an anti-self-dual connection! It is called the **standard instanton**, or the Belyavin-Polyakov-Schwarz-Tyupkin (BPST) instanton, after the physicists who discovered it in 1975.

This is a finite-energy Yang-Mills connection on $\mathbb{R}^4$. Dimension 4 is the only dimension for which Euclidean space can carry one.[21] In fact, this also can only work for nonabelian structure group. So the BPST instanton is truly a remarkable beast.

Let's now think about the second Chern class of this guy. We have

$$\begin{aligned}\operatorname{Tr}_{\mathbb{C}} F_A \wedge F_A &= \frac{\operatorname{Tr}}{(1+|x|^2)^4}(-2)^2\left(q_1^2(-2dV) + q_2^2(-2\,dV) + q_3^2(-2\,dV)\right)\\ &= \frac{1}{(1+|x|^2)^4}(-2)^4 3 = \frac{48}{(1+|x|^2)^4}dV.\end{aligned}$$

In polar coordinates, $dV = r^3 dr \wedge dV_{S^3}$; since the volume of S^3 is $2\pi^2$, integrating gives

$$\int_{\mathbb{R}^4} \operatorname{Tr}_{\mathbb{C}} F_A \wedge F_A = 96\pi^2 \int_0^\infty \frac{r^3\, dr}{(1+r^2)^4}.$$

Substitute $x = r^2$ and $dx = 2r\, dr$ to get

$$48\pi^2 \int_0^\infty \frac{(1+x-1)\, dx}{(1+x)^4} = 48\pi^2\left(\int_0^\infty \frac{1}{(1+x)^3}\, dx - \int_0^\infty \frac{1}{(1-x)^4}\, dx\right) = 8\pi^2.$$

Comparing this result with Example 16.8 suggests that A extends to a connection on an $\mathrm{SU}(2)$-bundle $E \to S^4$ with $c_2(E) = 1$.

[21]Note that the U(1)-connection we wrote down in Example 18.5 was Yang-Mills on S^2, but not on $\mathbb{R}^2$. In dimension four, the equations are conformally invariant, so our guy is actually ASD on both S^4 and $\mathbb{R}^4$.

This is true! Applying the gauge transformation $\sigma(x) = \frac{\bar{x}}{|x|}$ on $\mathbb{R}^4 \setminus \{0\}$, we can obtain a connection form that is smooth in the variable $y = x^{-1}$ (exercise). In fact, it is a general theorem of Uhlenbeck that any finite-energy Yang-Mills connection on $\mathbb{R}^4$ extends to S^4. We don't have time to discuss this result, unfortunately.

20.3. **Further discussion.** By applying translations and scaling to the standard instanton, we can write down a 5-dimensional family of instantons:

$$A_{\lambda,x_0}(x) = \frac{\operatorname{Im}((x - x_0)d\bar{x})}{\lambda^2 + |x - x_0|^2}.$$

(Exercise: show that these are instantons.) If you visualize, the parameter x_0 translates A around, and the λ scales A, making it more or less concentrated.

This story goes much, much further. It is a hard fact that the moduli space of instantons with $c_2 = 1$ on S^4 is homeomorphic to an open 5-ball, so this is all of them. The family can be compactified by adding in the S^4 on the boundary, and note that this is just our M. For more on this story, see Atiyah's book (about writing down all the instantons on S^4) and/or Donaldson & Kronheimer's book (about extracting invariants from compactified instanton moduli spaces on general 4-manifolds).

20.4. **Exercises.**

1. Show that $d\bar{x} \wedge dx$ is an $\operatorname{Im}\mathbb{H}$-valued self-dual 2-form on $\mathbb{R}^4$.
2. Prove that for $\alpha \in \Omega^1(\mathbb{H})$, $\operatorname{Im}(\alpha \wedge \alpha) = \operatorname{Im}(\alpha) \wedge \operatorname{Im}(\alpha)$.
3. Show that the standard instanton extends to a smooth connection on S^4.
4. Show that the 5-dimensional family above consists of instantons.

21. Uhlenbeck's Theorem: warm-up I (4/5-7)

Recall from §13 that a connection A satisfies $F_A \equiv 0$ if and only if there local exist frames τ such that $A^\tau \equiv 0$. We proved this just by choosing a "radial gauge," in which $A^\tau_r \equiv 0$, and deducing that $A^\tau \equiv 0$ if the curvature vanishes.

Uhlenbeck asked: suppose F_A is "small" (or maybe just bounded). Can you choose gauges such that A^τ is "small" (or bounded)? The answer is yes, in a very precise sense. The technique for showing this is called "Coulomb gauge fixing;" one tries to choose τ such that

(1) $d^*A^\tau = 0$
(2) A^τ is "as small as F_A."

As we'll see, in the right setup, the second item follows from the first, and is also used to prove it.

21.1. **Warm-up I: abelian case.** Let M be compact and $G = \mathrm{U}(1)$, so $\mathfrak{g}_E = i\underline{\mathbb{R}}$. Let d be the product connection on the trivial bundle, and write

$$\nabla_A = d + a$$

where $a \in \Omega^1_M(i\mathbb{R})$. Let

$$\sigma = \exp(\chi)$$

with $\chi \in i\mathbb{R}$. By our transformation law,

$$\sigma(A) = a - d\sigma\sigma^{-1} = a - d\chi.$$

The curvature is

$$F_A = da.$$

What is Coulomb gauge here? To make condition (1) hold, we need

$$d^*\sigma(A) = d^*a - d^*d\chi = d^*a - \Delta\chi = 0,$$

or equivalently,

$$\Delta\chi = d^*a.$$

This is solvable because

$$\int (d^*a)\, dV = \int a\, d(dV) = 0.$$

(This is the same fact we used above to construct Yang-Mills connections on line bundles over Riemann surfaces.) So a Coulomb gauge exists, establishing the first desideratum above.

Next, we note that the operator

$$d \oplus d^* : \Omega^1 \to \Omega^2 \oplus \Omega^0$$

is an example of an **elliptic operator**. We have

$$\begin{aligned} \|da\|^2_{L^2} + \|d^*a\|^2_{L^2} &= \int \left(|da|^2 + |d^*a|^2\right) dV \\ &= \int \left(\langle da, da\rangle + \langle d^*a, d^*a\rangle\right) dV \\ &= \int \left(\langle d^*da + dd^*a, a\rangle\right) dV \\ &= \int \langle \Delta_{Hodge} a, a\rangle\, dV. \end{aligned}$$

The Bochner-Weitzenböck formula says

$$\Delta_{Hodge} = \nabla^*\nabla + 0\text{'th order terms},$$

where ∇ is the Levi-Civita connection. Applying this yields

$$\int \langle \Delta_{Hodge} a, a\rangle\, dV = \int \langle \nabla^*\nabla a + (0\text{'th order})a, a\rangle\, dV \geq \int |\nabla a|^2 dV - C\int |a|^2\, dV.$$

Rearranging, we get

$$\int |a|^2 + \int |\nabla a|^2 \leq C\left(\int |da|^2 + \int |d^*a|^2 + \int |a|^2\right),$$

or in other words

$$\|a\|_{L^2_1} \leq C\left(\|da\| + \|d^*a\| + \|a\|\right), \tag{21.1}$$

where $\|\cdot\| = \|\cdot\|_{L^2}$ and $\|a\|_{L^2_1}$ is equal by definition to the LHS of (21.1). This is the simplest example of an **elliptic estimate**.

Claim. Since

$$H^1(M) = \ker\left(d \oplus d^*|_{\Omega^1}\right) = 0,$$

we may drop the $\|a\|$ term from the RHS (at the price of possibly increasing the constant), to obtain

$$\|a\|_{L^2_1} \leq C\left(\|da\| + \|d^*a\|\right). \tag{21.2}$$

Proof of claim. This follows from a well-known Rellich-plus-contradiction argument, which we now give.

Note that it is sufficient to show that

$$\|a\| \leq C\left(\|da\| + \|d^*a\|\right).$$

Assume that this fails for all constants C, and let a_n be a form for which the estimate fails with $C = n$. We may renormalize so that $\|a_n\| = 1$. Then we have

$$1 = \|a_n\|_{L^2} \geq n\left(\|da_n\| + \|d^*a_n\|\right).$$

Rearranging, we obtain

$$\|da_n\| + \|d^*a_n\| \leq \frac{1}{n}.$$

Clearly, from (21.1), the norms $\|a_n\|_{L^2_1}$ are uniformly bounded. Hence, by Banach-Alaoglu, we may take a weakly convergent subsequence

$$a_n \rightharpoonup b$$

in L^2_1. By Fatou's Lemma (lower-semicontinuity of norms under weak limits), we have

$$\|db\| + \|d^*b\| \leq \liminf_{n\to\infty} \|da_n\| + \|d^*a_n\| \leq \lim_{n\to\infty} \frac{1}{n} = 0.$$

But by Rellich, we have $a_n \to b$ strongly in L^2, so

$$\|b\|_{L^2} = \lim_{n\to\infty} \|a_n\|_{L^2} = \lim_{n\to\infty} 1 = 1.$$

Hence, b is a unit-norm element with $db = 0 = d^*b$. This contradicts the assumption that $\ker\left(d \oplus d^*|_{\Omega^1}\right) = 0$.

We conclude that with a large-enough constant, the desired estimate holds. ◇

If we're in Coulomb gauge, where $d^*a = 0$, then (21.2) reduces to

$$\|a\|_{L^2_1} \leq C\|F_A\|,$$

which is the second desideratum above.

21.2. **Sobolev spaces.** We'll now begin setting up the machinery to prove the existence of Coulomb gauges for nonabelian structure groups.

Let f be a function on a compact Riemannian n-manifold M. The **Sobolev norm** of f is

$$\|f\|_{L^p_k} \coloneqq \sum_{i=0}^{k} \left(\int_M |\nabla^{(i)} f|^p dV \right)^{1/p}. \tag{21.3}$$

The **Sobolev space**, also denoted L^p_k, is the completion of C^∞ with respect to the L^p_k norm.

Theorem 21.1 (Sobolev inequalies/embedding). *We have the following compact embeddings of Banach spaces.*

(1) $L^p_q \hookrightarrow L^{\frac{np}{n-p}}$ for $p < n$. In other words, the following Sobolev inequality holds:

$$\|f\|_{L^{\frac{np}{n-p}}} \le C \|f\|_{L^p_1}.$$

(2) More generally, $L^p_k \hookrightarrow L^q_\ell$ if

$$k - \frac{n}{p} \ge \ell - \frac{n}{q} \text{ and } k > \ell.$$

(3) $L^p_k \hookrightarrow C^0$ if $k - \frac{n}{p} > 0$.

Example 21.2.

- For $n \le 4$, we get $L^2_1 \hookrightarrow L^4$.
- If $n < 4$, then $L^2_2 \hookrightarrow C^0$, but this fails for $n = 4$.
- Generally, $L^p_2 \hookrightarrow C^0$ for $p > \frac{n}{2}$.

Lemma 21.3 (Sobolev multiplication I). *For $p \ge \frac{n}{2}$, multiplication of functions gives a smooth map of Banach spaces*

$$\begin{aligned} L^p_1 \times L^p_1 &\to L^p \\ (f, g) &\mapsto fg. \end{aligned}$$

Proof sketch. If $p \ge \frac{n}{2}$, then

$$L^p_1 \hookrightarrow L^{\frac{np}{n-p}} \hookrightarrow L^{2p}.$$

We have

$$\|fg\|_{L^p} = \left(\int f^p g^p \right)^{1/p} \le \left(\int f^{2p} \right)^{1/2p} \left(\int g^{2p} \right)^{1/2p} \le C \left(\|f\|_{L^p_1} \|g\|_{L^p_1} \right)$$

where the first inequality is by Hölder. Proving continuity/smoothness involves bounding polynomials in f and g using arguments similar to those above (exercise). □

Lemma 21.4 (Sobolev multiplication II). *For $p > \frac{n}{2}$, multiplication*

$$L^p_1 \times L^p_2 \to L^p_1$$

is continuous (indeed, smooth).

Proof. Since $2 - \frac{n}{p} > 0$,

$$L^p_2 \hookrightarrow C^0.$$

We have

$$\|\nabla(fg)\|_{L^p} = \|\nabla f \cdot g\|_{L^p} + \|f \cdot \nabla g\|_{L^p}.$$

Now,

$$\|\nabla f \cdot g\|_{L^p} \leq \|\nabla f\|_{L^p} \|g\|_{C^0} \leq \|\nabla f\|_{L^p} \|g\|_{L^p_2} \leq \|f\|_{L^p} \|g\|_{L^p_1}.$$

By the previous multiplication theorem,

$$\|f \cdot \nabla g\|_{L^p} \leq C \left(\|f\|_{L^p_1} \|\nabla g\|_{L^p_1} \right) \leq C \|f\|_{L^p_1} \|g\|_{L^p_2}.$$

Overall, we obtain

$$\|\nabla (f \cdot g)\|_{L^p} \leq C(\|f\|_{L^p_1} \|g\|_{L^p_2}).$$

Continuity follows by similar manipulations with the Leibniz rule. □

Lemma 21.5 (Sobolev multiplication III). *If $p > \frac{n}{2}$ then multiplication*

$$L^p_2 \times L^p_2 \to L^p_2$$

is continuous.

Proof. Exercise. □

Remark 21.6. Note that Sobolev multiplication III implies that L^p_2 is a Banach *algebra*, for $p > n/2$. These are very special objects.

21.3. **Sobolev sections, connections, and gauge transformations.** Fix a smooth reference connection ∇_{ref} on $E \to M$, coupled to ∇_{LC} when necessary. We can define Sobolev norms of differential forms and sections by the same rule (21.3), with ∇_{ref} in place of ∇. We obtain Sobolev spaces again as the completion of the space of smooth sections with respect to the norm. We will use an obvious notation in which, for example, the Sobolev space of L^p_k global adjoint-bundle-valued 1-forms will be denoted by

$$L^p_k \left(\Omega^1 \left(\mathfrak{g}_E \right) \right).$$

Note that a different choice of (smooth) reference connection would give a uniformly equivalent norm; so the topology of the Sobolev space is independent of the reference connection, and any estimates will depend on it only up to a constant.

Each of the above multiplication theorems applies to the natural operations that we know on these objects (wedge, tensor product, contraction, *etc.*). This follows just by remarking that these sections belong to the appropriate Sobolev spaces if and only if their local components do.

Now, in order to do gauge fixing for nonabelian structure groups, we also need Sobolev spaces of connections and gauge transformations. For any connection, we shall abuse notation by writing

$$\nabla_A = \nabla_{ref} + A,$$

with $A \in \Omega^1(\mathfrak{g}_E)$. This uniquely determines the global 1-form A in terms of the corresponding connection. We define the Sobolev norm of a connection to be the Sobolev norm of the corresponding 1-form.

Recall that we view the bundle of gauge transformations $\mathscr{G}_E \subset \mathrm{End} E$ as a subbundle of the endomorphism bundle. Hence, gauge transformations are in particular sections of $\mathrm{End} E$, and we may define the Sobolev norm of a gauge transformation to be its norm as a section of $\mathrm{End} E$.

Definition 21.7. Write $\mathscr{A}_k^p$ for the space of L_k^p Sobolev connections and $\mathscr{G}_{k+1}^p$ for the space of L_{k+1}^p Sobolev gauge transformations, with the latter viewed as a subset of $L_{k+1}^p(\mathrm{End} E)$.

Remark 21.8. If $\sigma \in \mathscr{G}_{k+1}^p$ with $p > \frac{n}{2}$ and $k \geq 1$, then $\det \sigma = 1$ and $|\sigma| = \sqrt{r}$, even after we take Sobolev completions. This is a crucial fact, requiring us to work with a compact structure group. The analysis below fails utterly for noncompact structure groups.

It is a remarkable fact that gauge theory "works" in the Sobolev setting.

Theorem 21.9.

(1) For $p \geq n/2$,

$$\begin{aligned} \mathscr{A}_1^p &\to L^p(\Omega^2(\mathfrak{g}_E)) \\ A &\mapsto F_A \end{aligned}$$

is continuous (smooth).

(2) For $p > n/2$, $\mathscr{G}_2^p$ forms a Banach Lie group with Lie algebra $L_2^p(\mathfrak{g}_E)$.

(3) For $p > n/2$, the gauge action

$$\begin{aligned} \mathscr{G}_2^p \times \mathscr{A}_1^p &\to \mathscr{A}_1^p \\ (\sigma, A) &\mapsto \sigma(A) \end{aligned}$$

is continuous (smooth).

Proof. (1) We have

$$F_A = F_{ref} + D_{ref} A + A \wedge A.$$

But $D_{ref} A \in L^p$, and by Sobolev multiplication I, $A \wedge A \in L^p$. So this map is continuous.

(2) The group property follows by the Sobolev multiplication Theorem III. Furthermore, since $L_2^p(\mathrm{End} E)$ is a Banach algebra, the exponential map is well-defined and smooth. Since σ takes values in $G \subset \mathrm{O}(r)$, we have $\det \sigma = 1$ and $|\sigma| = \sqrt{r}$, as remarked above. It follows from Cramer's rule and Sobolev multiplication III that the map $\sigma \mapsto \sigma^{-1}$ is continuous.

(3) We have

$$\sigma(A) = \sigma A \sigma^{-1} - D_{ref} \sigma \, \sigma^{-1}. \tag{21.4}$$

Since $\sigma \in L_2^p$, $A \in L_1^p$, and $\sigma^{-1} \in L_2^p$ by (2), the first term is in L_p^1 by Sobolev multiplication II. We also have $D_{ref} \sigma \in L_1^p$ by definition, so the second term also lives in L_1^p.

□

21.4. **Exercises.**

1. Convince yourself that the multiplication maps in the Sobolev embedding lemmas are smooth, as maps between Banach spaces.
2. Prove Sobolev multiplication III. (Hint: imitate the proof of II.)
3. Prove (21.4).

22. Uhlenbeck's Theorem: warm-up II, statement, key estimate (4/7-12)

22.1. **Warm-up II: Slice Theorem.** We'll construct a **relative Coulomb gauge**: if A and B are connections on E, then we want to transform B so that

$$D_A^*(\sigma(B) - A) = 0. \tag{22.1}$$

The motivation is as follows. Consider a family of gauge transformations

$$\sigma_t = \exp(-t\chi), \quad \chi \in \mathfrak{g}_E.$$

By Exercise 14.4.3., we have

$$\frac{d}{dt}\sigma_t(A) = D_A\chi.$$

Hence, the image of

$$D_A : \mathfrak{g}_E \to \Omega^1(\mathfrak{g}_E)$$

is that of the infinitesimal gauge action at A. Notice that

$$\operatorname{Ker}(D_A^*) = (\operatorname{Im} D_A)^\perp$$

because

$$\int \langle \alpha, D_A\chi \rangle = \int \langle D_A^*\alpha, \chi \rangle .$$

Hence, the space of connections in relative Coulomb gauge is an orthogonal "slice" of the gauge action. Slices are what give you coordinate charts on a quotient (in this case, the moduli space $\mathscr{B} = \mathscr{A}/\mathscr{G}$), so the following theorem is important in Donaldson theory.

Theorem 22.1 (Slice Theorem)**.** *Assume $p > n/2$. For all B sufficiently close to A in L_1^p, a relative Coulomb gauge $\sigma \in \mathscr{G}_2^p$, satisfying (22.1), exists.*

Proof. Write $B = A + a$, so

$$\begin{aligned} \sigma(B) &= \sigma\,(A + a)\,\sigma^{-1} - D_{ref}\sigma\,\sigma^{-1} \\ &= \sigma a \sigma^{-1} - (D_{ref}\sigma + A\sigma - \sigma A)\sigma^{-1} + A \\ \implies \sigma(B) - A &= \sigma a \sigma^{-1} - (D_A\sigma)\sigma^{-1} \end{aligned}$$

(recall that we define the 1-forms A and B by $D_A = D_{ref} + A$ and $D_B = D_{ref} + B$, respectively). Define a map

$$\begin{aligned} N : \mathscr{A}_1^p \times \mathscr{G}_2^p &\to L^p(\operatorname{Im} D_A^*) \\ (a, \sigma) &\mapsto D_A^*(-D_A\sigma\,\sigma^{-1} + \sigma a \sigma^{-1}). \end{aligned}$$

By Theorem 21.9, N is a smooth nonlinear map of Banach spaces.

Write $\sigma = \exp(\chi)$ for $\chi \in L_2^p(\mathfrak{g}_E)$. Then the linearization of N at $(0,0)$ is given by

$$L(a, \chi) = -D_A^* D_A \chi + D_A^* a.$$

As in the linear case,

$$D_A^* D_A : L_2^p(\mathfrak{g}_E) \to L^p(\operatorname{Im} D_A^*)$$

is onto because $D_A^* D_A$ is a (formally) self-adjoint elliptic operator.

(More precisely: since $D_A^* D_A$ is self-adjoint elliptic, its image is closed and its cokernel is equal to its kernel. In other words, $D_A^* D_A \chi = \alpha$ is solvable if and only if $\alpha \perp \operatorname{Ker} D_A^* D_A$. But $\operatorname{Ker} D_A^* D_A = \operatorname{Ker} D_A$, due to integration by parts. Hence, as above,

$$\alpha = D_A^* \beta \perp \operatorname{Ker} D_A = \operatorname{Ker} D_A^* D_A,$$

so we're done.)

By the implicit function theorem, there exists $\varepsilon > 0$ such that for all $\|a\|_{L_1^p} < \varepsilon$, there exists $\chi \in L_2^p$ with $N(a, e^\chi) = 0$. Then $\sigma = \exp(\chi)$ is the desired gauge. □

Note: This proof gives no useful control on ε (hence is just a warm-up); we want to get Coulomb gauge just assuming $\|F_A\|$ is small.

22.2. **Uhlenbeck's Theorem.** Let $d = D_{ref}$ be the product connection on $\underline{\mathbb{R}}^n = M \times \mathbb{R}^r$, where M is an n-dimensional compact manifold with $H^1(M) = 0$. For each $\varepsilon > 0$, let

$$\mathscr{A}_{1,\varepsilon}^p := \left\{ A \in \mathscr{A}_1^p \text{ on } \underline{\mathbb{R}}^r : \|F_A\|_{L^{n/2}} \le \varepsilon \right\}.$$

Let $\mathscr{A}_\varepsilon \subset \mathscr{A}_{1,\varepsilon}^p$ be the path component of d inside $\mathscr{A}_{1,\varepsilon}^p$. *I.e.*, $\mathscr{A}_\varepsilon$ is the set of connections with small curvature in $L^{n/2}$ that are connected to the trivial connection by a path of such connections.

Theorem 22.2 ("Uhlenbeck's Theorem[22]"). *Assume $\frac{n}{2} < p < n$ and $H^1(M) = 0$. For $\varepsilon > 0$ sufficiently small (depending on M, r), the following hold: for all $A \in \mathscr{A}_\varepsilon$ there exists $\sigma \in \mathscr{G}_2^p$ such that*

(1) $d^* \sigma(A) = 0$
(2) $\|\sigma(A)\|_{L_1^q} \le C \|F_A\|_{L^q}$ for all $n/2 \le q \le p$.

The proof is contained in the next subsection. We will use the "method of continuity:" given a path A_t from $A_0 = d$ to $A_1 = A$, with $A_t \in \mathscr{A}_{1,\varepsilon}^p$ for each $t \in [0,1]$, we will show that

$$\{t \in [0,1] : A_t \text{ satisfies (1) and (2)}\}$$

is both open and closed. The particular framework of the proof is not as important as the following *a priori* estimate, on which it depends crucially.

Lemma 22.3 (Key estimate). *There exists $\eta > 0$ and $N > 0$ (depending only on M, r, and p) such that if*

$$d^* A = 0$$

and

$$\|A\|_{L^n} < \eta,$$

then

$$\|A\|_{L_1^q} \le N \|F_A\|_{L^q}$$

for all $\frac{n}{2} \le q \le p$.

[22]Uhlenbeck proved the result not on a compact manifold but on a ball with Neumann boundary conditions; this is a "lazy" version of her result, which is sufficient for our purposes. The textbook "Uhlenbeck Compactness" by Wehrheim contains a full exposition of the original version of Uhlenbeck's Theorem.

Proof. We have the following elliptic estimate:

$$\|A\|_{L^q_1} \le C_1(\|dA\|_{L^q} + \|d^*A\|_{L^q} + \|A\|_{L^q}).$$

We proved the $q = 2$ case yesterday; see Donaldson and Kronheimer, Appendix A.V (and references therein), for the general case. By the same argument as for $q = 2$, under the assumption $H^1(M) = 0$, we can drop the $\|A\|_{L^q}$ term from the RHS:

$$\|A\|_{L^q_1} \le C_1\left(\|dA\|_{L^q} + \|d^*A\|_{L^q}\right). \tag{22.2}$$

We first prove a special case where the numbers are simpler, then do the general case.

The case $n = 4, q = 2$. Here we have the Sobolev inequality

$$\|A\|_{L^4} \le C_2\|A\|_{L^2_1}.$$

Combining this with the above elliptic estimate, we obtain

$$\begin{aligned}\|A\|_{L^4} &\le C_1C_2\left(\|dA\| + \|d^*A\|\right)\\ &\le C_1C_2\,\|dA\|,\end{aligned}$$

by the Coulomb gauge condition. Here as always we write $\|\cdot\| = \|\cdot\|_{L^2}$.

Recall that $F_A = dA + A\wedge A$. We estimate the quadratic term in the curvature, as follows:

$$\begin{aligned}\|A\wedge A\|_{L^2} \le \|A\|^2_{L^4} &\le \|A\|_{L^4}\left(C_1C_2\,\|dA\|\right)\\ &\le C_1C_2\eta\,\|dA\|.\end{aligned}$$

Since $dA = F_A - A\wedge A$, we may rewrite the elliptic estimate above using the triangle inequality:

$$\|A\|_{L^2_1} \le C_1\,\|dA\| \le C_1(\|F_A\| + \|A\wedge A\|) \le C_1\,\|F_A\| + C_1C_2\eta\,\|A\|.$$

We now assume that

$$\eta \le \frac{1}{2C_1C_2}. \tag{22.3}$$

Then we may rearrange the last inequality, to obtain

$$\frac{1}{2}\,\|A\|_{L^2_1} \le (1 - C_1C_2\eta)\,\|A\|_{L^2_1} \le C_1\,\|F_A\|.$$

In summary,

$$\|A\|_{L^2_1} \le 2C_1\,\|F_A\|.$$

Hence, for η as in (22.3), the desired inequality holds with $N = 2C_1$.

The general case. We shall let $q = p$, since the proof is the same for q in the stated range. We have

$$\|A\wedge A\|_{L^p} \le \|A\|^2_{L^{2p}}$$

Since $p < n$, we have the Sobolev inequality[23]

$$\|A\|_{L^{p'}} \le C\,\|A\|_{L^p_1},$$

[23] We will now follow the standard convention of using the same letter C for the constant in each estimate, where C is allowed to increase as the text goes along. To make things kosher, one would just go through the text in order and put a label on each C. Hence, as before, our abuse of notation is no more than suppression of labels.

where

$$1 - \frac{n}{p} = -\frac{n}{p'}. \tag{22.4}$$

We also have the interpolation inequality (really just Hölder's inequality)

$$\|A\|_{L^{2p}} \leq \|A\|_{L^{p'}}^{1/2} \|A\|_{L^{p''}}^{1/2},$$

where p'' is determined by

$$\frac{1}{2p} = \frac{1}{2p'} + \frac{1}{2p''}.$$

Multiplying both sides by $2n$ and substituting (22.4) above, we have

$$\frac{n}{p} = \frac{n}{p} - 1 + \frac{n}{p''}.$$

This simplifies to

$$p'' = n.$$

Combining the Sobolev and interpolation inequalities above, we obtain

$$\|A \wedge A\|_{L^p} \leq \|A\|_{L^{2p}}^2 \leq \|A\|_{L^n} \|A\|_{L^{p'}} \leq C \|A\|_{L^n} \|A\|_{L_1^p}. \tag{22.5}$$

Applying the Coulomb gauge condition in the elliptic estimate (22.2), we have

$$\begin{aligned} \|A\|_{L_1^p} \leq C \|dA\|_{L^p} &\leq C(\|F_A\|_{L^p} + \|A \wedge A\|_{L^p}) \\ &\leq C(\|F_A\|_{L^p} + \|A\|_{L^n} \|A\|_{L_1^p}) \\ &\leq C(\|F_A\|_{L^p} + \eta \|A\|_{L_1^p}). \end{aligned}$$

As long as $\eta \leq \frac{1}{2C}$, we can absorb the rightmost term (as above) to obtain the desired estimate. □

22.3. **Exercises.**

1. Show that B is in relative Coulomb gauge with respect to A if and only if A is in relative Coulomb gauge with respect to B.

23. Uhlenbeck's Theorem: proof, statement of Compactness Theorem (4/12-14)

We're now in a position to prove Theorem 22.2.

Given $A \in \mathscr{A}_\varepsilon$, let $A_t \in \mathscr{A}_{1,\varepsilon}^p$ be a path with $A_0 = 0$ and $A_1 = A$. Let $S \subset [0,1]$ be the set of t for which there exists $\sigma \in \mathscr{G}_2^p$ such that

$$d^* \sigma(A_t) = 0 \tag{U1}$$

and

$$\|\sigma(A_t)\|_{L_1^q} \leq N \|F_A\|_{L^q} \tag{U2}$$

for $n/2 \leq q \leq p$, where N is the constant of the Key Estimate above. Note that $0 \in S$, since A_0 is the product connection, which satisfies (U1-U2) trivially. Hence S is nonempty. Since $[0,1]$ is connected, it suffices to show that S is both open and closed.

Openness. Fix $t_0 \in S$ and let σ_0 be the gauge transformation such that (U1-U2) hold for $B = \sigma_0(A_{t_0})$.

First of all, note that the Sobolev conjugate of $n/2$ is

$$\left(\frac{n}{2}\right)^* = \frac{n(n/2)}{n - n/2} = n.$$

This gives us

$$\|B\|_{L^n} \leq C\|B\|_{L_1^{n/2}} \leq CN\|F_B\|_{L^{n/2}} \leq CN\varepsilon.$$

Hence, as long as we assume

$$\varepsilon < \frac{\eta}{CN},$$

we have

$$\|B\|_{L^n} < \eta. \tag{U3}$$

Now, given a further gauge transformation $\sigma = \exp(\chi)$, we have

$$d^*(\sigma(B + b_t)) = d^*(\exp(\chi)(B + b_t)\exp(-\chi) - d\exp(\chi)\exp(-\chi).$$

As before, the linearization is

$$d^*(-D_B\chi + b_t).$$

Claim. With B as above, d^*D_B is surjective onto $(\operatorname{Ker} d)^\perp = \operatorname{Im} d^* \subset \mathfrak{g}_E$.

Proof of claim. [24] Using the formula (17.4) for the adjoint and the Coulomb gauge condition, we may write

$$\begin{aligned} d^*D_B\chi &= d^*d\chi + d^*[B,\chi] \\ &= d^*d\chi + [d^*B,\chi] - [B.d\chi] \\ &= d^*d\chi - [B.d\chi]. \end{aligned}$$

(Exercise: check this manipulation carefully.) Using the triangle inequality and the same Hölder inequality as in (22.5) above, we obtain

$$\|d^*D_B\chi\|_{L^p} \geq \|d^*d\chi\|_{L^p} - \|B\|_{L^n}\|d\chi\|_{L^{p'}}.$$

Now, since $dd\chi = 0$, our elliptic estimate (22.2) reads

$$C\|d^*d\chi\|_{L^p} \geq \|d\chi\|_{L_1^{p'}}.$$

The Sobolev inequality gives

$$C\|d\chi\|_{L_1^p} \geq \|d\chi\|_{L^{p'}}.$$

Combining these with (U3), we obtain

$$\begin{aligned} \|d^*D_B\chi\|_{L^p} &\geq \frac{1}{C}\|d\chi\|_{L_1^p} - C\|B\|_{L^n}\|d\chi\|_{L_1^p} \\ &\geq \left(\frac{1}{C} - C\varepsilon\right)\|d\chi\|_{L_1^p}. \end{aligned}$$

After possibly taking ε smaller, this implies the claim. ◇

[24] Omitted during class.

Now, write $\sigma_0(A_t) = B + b_t$, so b_t is a path in $\Omega^1(\mathfrak{g}_E)$ with $b_{t_0} = 0$. By the implicit function theorem, for t sufficiently close to t_0, there is a smooth family $\chi_t \in \mathfrak{g}_E$, with $\chi_{t_0} = 0$, such that

$$d^*(\exp(\chi_t)(B + b_t)) = 0.$$

Letting $\sigma_t = \exp(\chi_t) \cdot \sigma_0$, we have

$$d^*\sigma_t(A_t) = 0,$$

which is (U1).

It remains to establish (U2) for t in a neighborhood of t_0. But (U2) implies (U3), which is an open condition; so for t sufficiently close to t_0, we still have

$$\|\sigma_t(A_t)\|_{L^n} < \eta.$$

By the Key Estimate, we therefore have

$$\|\sigma_t(A_t)\|_{L^q_1} \leq N \|F_{A_t}\|_{L^q}$$

for $n/2 \leq q \leq p$. This establishes (U2), completing the proof that S is open.

Closedness. Suppose that $t_i \in S$ and $t_i \to t_\infty \in [0,1]$. We need to prove that $t_\infty \in S$.

Our setup provides $A_i \coloneqq A_{t_i} \to A_{t_\infty} =: A_\infty \in \mathscr{A}^p_1$, for which there are σ_i such that

$$A'_i \coloneqq \sigma_i(A_i) = \sigma_i A_i \sigma_i^{-1} - d\sigma_i\, \sigma_i^{-1}. \tag{23.1}$$

satisfy (U1) and (U2). Both of these conditions are intuitively "closed conditions": (U1) is an equation, and both sides of (U2) are continuous functions of the connection. So if we were able to extract strong limits of $\sigma_i \in \mathscr{G}^p_2$ and $A'_i \in \mathscr{A}^p_1$, then we'd be done. In fact, we'll only be able to extract weak limits, but this will be sufficient.

We define the weak topology based on the inclusions

$$\mathscr{G}^p_2 \subset L^p_2(\mathrm{End} E)$$

and the identification

$$\mathscr{A}^p_1 \cong L^p_1(\Omega^1(\mathfrak{g}_E)),$$

which (as above) is based on fixing a product connection $d = D_{ref}$ as our reference connection. Note that L^p_k is a reflexive Banach space with dual $L^{\frac{p}{p-1}}_{-k}$, so the weak topology is defined by integration of the section and its derivatives against $L^{\frac{p}{p-1}}$ sections.

Since each A'_i satisfies (U2) by assumption, this sequence is bounded in L^p_1. Hence, by the Banach-Alaoglu Theorem, we may assume that $A'_i \rightharpoonup A'_\infty$ in L^1_p after passing to a subsequence.

We will now construct a gauge transformation $\sigma_\infty \in \mathscr{G}^p_2$ such that $\sigma_\infty(A_\infty) = A'_\infty$. The technique for doing this is called "bootstrapping."

We first multiply (23.1) by σ_i, to obtain

$$d\sigma_i = \sigma_i A_i - A'_i \sigma_i. \tag{23.2}$$

Since $\sigma_i \in L^\infty$ and $A_i, A'_i \in L^p_1 \subset L^{p'}$, we have

$$\sup_i \|d\sigma_i\|_{L^{p'}} < \infty.$$

Hence, again passing to a subsequence (if necessary), we may assume $\sigma_i \rightharpoonup \sigma_\infty$ in $L^{p'}_1$.

Now, differentiating gives

$$\nabla d\sigma_i = \nabla\sigma_i A_i + \sigma_i \cdot \nabla A_i - \nabla A_i'\sigma_i - A_i'\nabla(\sigma_i)^{-1}.$$

We have uniform bounds on $\nabla\sigma_i \in L^{p'}$, $A_i \in L^{p'}$, $\sigma_i \in L^\infty$, $\nabla A_i \in L^p$, $\nabla A_i' \in L^p$, $A_i' \in L^{p'}$, and $\nabla(\sigma_i)^{-1} \in L^{p'}$. Note that

$$p' = \frac{np}{n-p} > \frac{np}{n/2} = 2p,$$

so multiplication

$$L^{p'} \times L^{p'} \to L^p$$

is continuous. Putting these all together, we have

$$\sup_i \|\nabla d\sigma_i\|_{L^p} < \infty,$$

i.e., σ_i is bounded in L^p_2. After again passing to a subsequence, we may assume that $\sigma_i \rightharpoonup \sigma_\infty$ weakly in L^p_2. This completes our "bootstrap."

We will now argue that the equation

$$d\sigma_\infty = \sigma_\infty A_\infty - A_\infty'\sigma_\infty \tag{23.3}$$

holds. We know that all the terms belong to L^p_1, based on Theorem 21.9, but it remains to check that (23.2) implies (23.3) under weak convergence $\sigma_i \rightharpoonup \sigma_\infty$, $A_i \rightharpoonup A_\infty$, and $A_i' \rightharpoonup A_\infty'$.

Since σ_i converge weakly in L^p_2, $d\sigma_i \rightharpoonup d\sigma_\infty$ weakly in L^p_1, which takes care of the LHS. Since A_i converges weakly in L^p_1, the product $\sigma_i A_i$ converges weakly to $\sigma_\infty A_\infty$ in L^p_1; this can be checked using a standard trick and the definition of weak convergence based on integration against functions in $L^{\frac{p-1}{p}}$ (exercise). Similarly, $A_i'\sigma_i$ converges weakly to $A_\infty\sigma_\infty$. Hence, each term of (23.2) converges L^p_1-weakly to the corresponding term in (23.3); since weak limits are unique, we conclude that (23.3) holds, as desired. It follows that $\sigma_\infty(A_\infty) = A_\infty'$.

Finally, we must show that $\sigma_\infty(A_\infty)$ satisfies (U1-U2). Since d^* is continuous from $L^p_1 \to L^p$, the equation (U1) is preserved under weak limits. For (U2), we argue as follows. By Fatou's Lemma (*i.e.* lower-semicontinuity of the norm under a weak limit), we have

$$\|\sigma_\infty(A_\infty)\|_{L^p_1} = \|A_\infty'\|_{L^p_1} \le \liminf_{i\to\infty} \|\sigma_i(A_i)\|_{L^p_1}.$$

From (U2), we have

$$\liminf_{i\to\infty} \|\sigma_i(A_i)\|_{L^p_1} \le C \liminf_{i\to\infty} \|F_{A_i}\|_{L^p}.$$

But since $A_i \to A_\infty$ strongly, we have

$$\liminf_{i\to\infty} \|F_{A_i}\|_{L^p} = \lim_{i\to\infty} \|F_{A_i}\|_{L^p} = \|F_{A_\infty}\|_{L^p}.$$

And, in view of the gauge equivalence (23.3), we know that

$$|F_{A_\infty}| = |\sigma_\infty F_{A_\infty}\sigma_\infty^{-1}| = |F_{\sigma_\infty(A_\infty)}|$$

almost-everywhere. Hence

$$\|F_{A_\infty}\|_{L^p} = \|F_{\sigma_\infty(A_\infty)}\|_{L^p}.$$

Combining these observations, we obtain (U2) for $\sigma_\infty(A_\infty)$. This shows that S is closed, completing the proof. □

Corollary 23.1. *Fix $n/2 < p < n$. Let A be a connection over the unit ball $B_1 \subset \mathbb{R}^n$ with*

$$\|F_A\|_{L^{n/2}} < \varepsilon_0.$$

Then there exists a gauge transformation σ such that

$$\|\sigma(A)\|_{L^q_1(B_{1/2})} \le C\,\|F_A\|_{L^q(B_1)}$$

for $n/2 \le q \le p$

Proof. Choose a smooth map $\varphi : S^n \to B_1$ such that the upper hemisphere S^n_+ is mapped diffeomorphically onto $B_{3/4} \subset B_1$. Let $D = \varphi^{-1}(B_{1/2}) \cap S^n_+$.

Given a connection A on B_1, form the pullback $\varphi^* A$ on S^n. Since φ is a smooth map from a compact space, $|d\varphi|$ is bounded, so

$$|F_{\varphi^* A}(x)| = |\varphi^* F_A(x)| \le C|F_A(\varphi(x))|.$$

We now obtain

$$\|F_{\varphi^* A}\|_{L^{n/2}(S^n)} \le C\,\|F_A\|_{L^{n/2}(B_1)}\,.$$

For ε_0 sufficiently small,

$$\|F_{\varphi^* A}\|_{L^{n/2}(S^n)} \le C\varepsilon_0 \le \varepsilon,$$

where ε is the constant from Uhlenbeck's theorem. If σ' is a Coulomb gauge on S^n, then

$$\sigma = (\varphi|_D^{-1})^* \sigma'$$

is the desired gauge on $B_{1/2}$.

It remains to show that $\varphi^* A \in \mathscr{A}_\varepsilon$, the connected component of d in $\mathscr{A}^p_{1,\varepsilon}$. Consider the map

$$\begin{aligned} \psi_t : B_1 &\to B_t \subset B_1 \\ x &\mapsto tx \end{aligned}$$

for $0 \le t \le 1$. We have

$$\begin{aligned} \psi_t^* A &= tA(t \cdot x) \\ \psi_0^* A &= 0 \\ \psi_1^* A &= A \end{aligned}$$

and

$$F_{\psi_t^* A}(x) = \psi_t^* F_A(x) = t^2(dA(tx) + (A \wedge A)(tx)) = t^2 F_A(tx).$$

Also,

$$\begin{aligned} \left\|F_{\psi_t^* A}\right\|^{n/2}_{L^{n/2}} &= \int_{B_1} |t^2 F_A(tx)|^{n/2}\, dV \\ &= \int_{B_1} |F_A(tx)|^n t^n\, dV \\ &= \int_{B_t} |F_A(x)|^n\, dV \\ &= \|F_A\|^{n/2}_{L^{n/2}(B_t)} \\ &\le \|F_A\|^{n/2}_{L^{n/2}(B_1)} < \varepsilon_0. \end{aligned}$$

This allows us to apply Uhlenbeck's theorem to $\varphi^*\psi_t^* A_t$. □

Note that Hölder's inequality gives us

$$\|F_A\|_{L^{n/2}(B_r)} \le r^{1-n/4}\|F_A\|_{L^2(B_r)} \le 2r^{1-n/4}\mathcal{YM}(A).$$

Hence, if $n \le 4$, the Yang-Mills energy controls the $L^{n/2}$ norm of the curvature, with an improvement on small balls for $n < 4$. This observation leads to:

Theorem 23.2 (Uhlenbeck's compactness theorem, low-dimensional version)**.** *If M is a closed manifold of dimension $n \le 3$, then any sequence of connections $\{A_i\} \subset \mathscr{A}_1^p$, $n/2 < p < n$, with uniformly bounded Yang-Mills energy, has a weakly convergent subsequence* **modulo gauge***, which converges strongly in $\mathscr{A}_1^q$ for any $q < p$.*

Proof sketch. Pick neighborhoods on which to apply Uhlenbeck's Theorem. You can bound how small they have to be using the uniform bound on energy. Construct a good gauge on each neighborhood using the corollary, then patch them together. (This is nontrivial, but we don't have time to discuss it.) □

23.1. **Exercises.**

1. Read the proof of the surjectivity claim in the openness part of the proof, and make sure you believe the calculation.
2. Prove that if $f_i \rightharpoonup f$ in L_2^p and $g_i \rightharpoonup g$ in L_1^p, where $p > n/2$, then
$$f_i g_i \rightharpoonup fg$$
in L_1^p. (Hint: use the trick $f_ig_i - fg = (f_i - f)g_i - f(g_i - g)$ together with the definition of weak convergence by integration.)
3. Make sure you're convinced by the ending of the closedness proof.

Part IV. Holomorphic bundles

24. Definition and first examples (4/19)

I'll assume that you've seen this material and go fast. If not, you can look at my complex manifolds notes.

24.1. **The definition.** Let M be a complex manifold with $\dim_{\mathbb{C}} M = n$. This means that we have coordinate charts $\{z^i\} \subset \mathbb{C}^n$ on M with underlying real charts

$$\{x^1, y^1, \ldots, x^n, y^n\} \subset \mathbb{R}^{2n}$$

with $z^i = x^i + \sqrt{-1}y^i$. For $\{w^i\} \subset \mathbb{C}^n$ any other complex chart, the transition map $z^i(w^j)$ is required to be holomorphic.

The **complex Jacobian** of a transition map is the $n \times n$ complex matrix-valued function

$$\left(\frac{\partial z^i}{\partial w^j}\right)_{ij}.$$

Here we use the standard notations

$$dz^i = dx^i + \sqrt{-1}dy^i \qquad \frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial y^i}\right)$$

$$d\bar{z}^i = dx^i - \sqrt{-1}dy^i \qquad \frac{\partial}{\partial \bar{z}^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial y^i}\right).$$

If we think of the complex Jacobian as a $2n \times 2n$ matrix over $\mathbb{R}$, then it agrees with the real Jacobian. (This is true if and only if a map is holomorphic.)

Let $\pi : E \to M$ be a smooth complex vector bundle of (complex) rank r.

Definition 24.1. A **holomorphic structure** $\mathcal{E}$ on E is a complex manifold structure on the total space E such that $\pi : E \to M$ is a holomorphic submersion. In particular, E has **local holomorphic frames**; that is, frames (in the usual sense) that consist of holomorphic sections

$$\{f_\mu : M \to E\}_{\mu=1}^r.$$

Equivalently, for any local holomorphic section $s : U \subset M \to E$, there are holomorphic functions s^μ, $\mu = 1, \ldots, r$, such that

$$s = s^\mu f_\mu.$$

The transition functions are defined as usual: if $\{f^a_\mu\}$ and $\{f^b_\nu\}$ are frames, then ${g_{ab}}^\nu{}_\mu$ are defined by

$$f^a_\mu = {g_{ab}}^\nu{}_\mu f^b_\nu.$$

Now ${g_{ab}}^\nu{}_\mu$ are holomorphic functions satisfying the cocycle conditions. Conversely, given any collection $\{{g_{ab}}^\mu{}_\nu\}$ of holomorphic matrix-valued functions satisfying the cocycle conditions, there exists a unique holomorphic vector bundle with these transition functions. This is shown exactly as in the continuous case. Consequently,

$$\{\text{holomorphic vb's on } M \text{ of rank } r\}/\text{isomorphism} \cong H^1(M, \mathscr{GL}(r, \mathbb{C})),$$

where $\mathscr{GL}(r, \mathbb{C})$ is the sheaf of holomorphic $\mathrm{GL}(r, \mathbb{C})$-valued functions.

In the rank 1 case, this is actually a useful description:

$$\mathrm{Pic}(M) := \{\text{holomorphic line bundles}\}/\text{isomorphism} \cong H^1(M, \mathscr{O}_M^*).$$

The **holomorphic exponential sequence** is the short exact sequence of sheaves

$$0 \to \underline{\mathbb{Z}} \xrightarrow{2\pi i \cdot} \mathscr{O}_M \xrightarrow{\exp(\cdot)} \mathscr{O}_M^* \to 0.$$

Note that this is indeed an exact sequence, since in the sheaf category one only requires surjectivity on stalks (exercise). The associated long exact sequence has a segment

$$\cdots \to H^1(M, \underline{\mathbb{Z}}) \to H^1(M, \mathscr{O}_M) \to H^1(M, \mathscr{O}_M^*) = \mathrm{Pic}(M) \xrightarrow{c_1} H^2(M, \underline{\mathbb{Z}}) \to \cdots.$$

Hence, we should think of $\mathrm{Pic}(M)$ as having discrete part, coming from $H^2(M, \underline{\mathbb{Z}})$ and a continuous part coming from $H^1(M, \mathscr{O}_M)$. If M is a Riemann surface of genus g, then $H^1(M, \mathscr{O}_M) \cong \mathbb{C}^g$, $H^1(M, \mathbb{Z}) \cong \mathbb{Z}^{2g}$, $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$, and $H^2(M, \mathscr{O}_M) = 0$, so

$$\mathrm{Pic}(M) \cong \mathbb{C}^g / \mathbb{Z}^{2g} \oplus \mathbb{Z}.$$

Here the embedding of $\mathbb{Z}^{2g}$ in $\mathbb{C}^g$ need not be the standard one.

"All" bundle operations work in the holomorphic category (barring conjugation, of course). However, there are a few major differences:

- Exact sequences do not always split.
- The Homotopy Theorem fails (you should expect this because Pic has a continuous part).
- The space of global sections of a holomorphic bundle is a finite-dimensional complex vector space (provided M is compact).

24.2. **Examples.**

1. $\mathscr{O}(k) \to \mathbb{CP}^n$.
2. The **holomorphic tangent bundle** $T^{1,0}M$ of any complex manifold M. This has transition functions $\left\{\frac{\partial z^i}{\partial w^j}\right\}$. As a real bundle, it is isomorphic to the tangent bundle of the smooth manifold M. (One can see this either by defining a canonical isomorphism between real and holomorphic tangent vectors at a point, or by remarking that the underlying real matrix of the holomorphic Jacobian agrees with the real Jacobian. For either point of view, see my complex manifolds notes).
3. The **holomorphic cotangent bundle** (or rather its sheaf of sections) is sometimes denoted by
$$\Omega_M = \Gamma_{hol}(T^*M).$$
This is the bundle/sheaf of holomorphic 1-forms on M.
4. The **canonical (line) bundle** $K_M := \wedge^n \Omega_M$.
5. Consider the rank-2 bundle $\mathscr{O}(1) \oplus \mathscr{O}(-1) \to \mathbb{CP}^1$. This has a holomorphic section vanishing at an isolated point. Meanwhile, there is an exact sequence
$$0 \to \mathscr{O}(-1) \to \underline{\mathbb{C}}^2 \to \mathscr{O}(1) \to 0.$$
However, $\underline{\mathbb{C}}^2$ has no section vanishing at an isolated point. Therefore
$$\underline{\mathbb{C}}^2 \ncong \mathscr{O}(-1) \oplus \mathscr{O}(1)$$

holomorphically; this is an example of a non-split exact sequence of holomorphic bundles.

24.3. **Line bundles associated to divisors.** Let $M = \Sigma$ be a compact Riemann surface, *i.e.* a compact complex manifold of complex dimension one.

Fix $p \in \Sigma$ and let $U_1 \ni p$ be a coordinate chart with

$$U_1 \cong D_1 = \{z : |z| < 1\} \subset \mathbb{C}.$$

Let

$$U_0 = \Sigma \setminus \{p\}.$$

The **point bundle** $\mathscr{O}(p)$ is by definition the holomorphic vector bundle with transition function

$$g_{10} = z^{-1}$$

on $U_0 \cap U_1 \cong D_1 \setminus \{0\}$.

Let $s^1 = z$ on U_1 and $s^0 = 1$ of U_1. This defines a global holomorphic section, s, of $\mathscr{O}(p)$, because

$$s^0 = 1 = z^{-1} z = g_{10} s^1.$$

Note that s vanishes transversely at p.

IMPORTANT WARNING. The notation $\mathscr{O}(p)$, with $p \in \Sigma$ a point, should not be confused with $\mathscr{O}(k)$, with $k \in \mathbb{Z}$ an integer. The former is a point bundle over an arbitrary Riemann surface, whereas the latter is a power of the hyperplane bundle over $\mathbb{CP}^n$. It is of course true that for $\Sigma = \mathbb{CP}^1$ and any point p, $\mathscr{O}(p) \cong \mathscr{O}(1)$ (exercise).

Claim. $\deg(\mathscr{O}(p)) = 1$

First proof of claim. Let N, S be the north and south poles of $\mathbb{CP}^1$. We have $\mathscr{O}_{\mathbb{CP}^1}(N) \cong \mathscr{O}_{\mathbb{CP}^1}(1)$. A classifying map for $\mathscr{O}(p) \to \Sigma$ is obtained by sending $D_{1/2} \subset U_1$ to $\mathbb{CP}^1 \setminus \{S\}$ homeomorphically, and crushing $\Sigma \setminus U_1$ to S. Based on the transition functions, this is clearly covered by a bundle map $\mathscr{O}(p) \to \mathscr{O}(1)$. Since this is a degree 1 map, we get the result. □

Second proof of claim. Choose a smooth cutoff function φ on Σ with

$$\varphi = \begin{cases} 0 \text{ on } D_{1/2} \subset U_1 \\ 1 \text{ on } \Sigma \setminus D_{3/4} \subset U_0. \end{cases}$$

Define a connection on $\mathscr{O}(p)$ as follows:

$$\begin{aligned} \text{On } U_1: &\quad A^1 = -\varphi dz\, z^{-1} \\ \text{On } U_0: &\quad A^0 = (1-\varphi) dz\, z^{-1}. \end{aligned}$$

The corresponding curvature form is

$$F_A = -d(\varphi d \log z).$$

Since $F_A|_{\Sigma \smallsetminus D_{3/4}} = d^2 \log z \equiv 0$, we have

$$\begin{aligned}
\deg(\mathscr{O}(p)) &= \frac{i}{2\pi}\int_M F_A \\
&= -\frac{i}{2\pi}\int_{U_1} d(\varphi d\log z) \\
&= \frac{-i}{2\pi}\int_{S^1} d\log z \\
&= -\frac{i}{2\pi} 2\pi i = 1.
\end{aligned} \tag{24.1}$$

(Exercise: why doesn't this calculation violate Stokes's Theorem?) □

We note that this is an example of the general fact that the degree of a holomorphic bundle is given by the number of zeroes minus the number of poles (counted with multiplicity) of a meromorphic section.

Definition 24.2. A **divisor** on Σ is a formal $\mathbb{Z}$-linear combination of points of Σ. Given a divisor

$$D = \sum_i n_i p_i - \sum_j m_j q_j,$$

where $n_i, m_j > 0$, set

$$\mathscr{O}(D) := \bigotimes_i \mathscr{O}(p_i)^{\otimes n_i} \otimes \bigotimes_j (\mathscr{O}(q_j)^{\otimes m_i})^*.$$

It follows from the additive property of the first Chern class (Lemma 8.4) and the above claim for point bundles that

$$\deg \mathscr{O}(D) = \sum n_i - \sum m_j.$$

Alternatively, we can define $\mathscr{O}(D)$ by transition functions, as above: let $U_i \ni p_i$ be disjoint small balls, $U_0 = \Sigma \smallsetminus \{p_i\}$, and take $g_{i0} = z^{-n_i}$. We have the following well-known correspondence:

$$\{\text{Global holomorphic sections of } \mathscr{O}(D)\} \cong \left\{\begin{array}{r}\text{Global meromorphic functions on } \Sigma \\ \text{with poles of order} \le n_i \text{ at } p_i \\ \text{and zeroes of order} \ge m_i \text{ at } q_i\end{array}\right\}. \tag{24.2}$$

In our setup, the correspondence is simply gotten by taking a global section s to its local component s^0 on the chart $U_0 = \Sigma \smallsetminus (\{p_i\} \cup \{q_i\})$.

As an application, we can show the following:

Claim. If Σ is compact with genus ≥ 1, then $\mathscr{O}(p) \ncong \mathscr{O}(q)$ for $p \ne q$.

Proof of claim.

$$\mathrm{Hom}(\mathscr{O}(q), \mathscr{O}(p)) \cong \mathscr{O}(p) \otimes \mathscr{O}(q)^* \cong \mathscr{O}(p-q).$$

A nonzero section of the RHS is a meromorphic function with a single simple pole—equivalently, a non-constant, degree 1 map $\Sigma \to \mathbb{CP}^1$. Since degree 1 holomorphic maps are invertible, this cannot exist. □

Note that by contrast, we must have

$$\mathscr{O}(p) \cong_{C^\infty} \mathscr{O}(q)$$

for any p, q, if Σ is connected, since a complex line bundle is determined up to smooth isomorphism by its degree (Theorem 8.3).

24.4. **Exercises.**

1. Prove that the holomorphic exponential sequence is an exact sequence of sheaves.
2. Show that for any point $p \in \mathbb{CP}^1$, we have $\mathscr{O}(p) \cong \mathscr{O}(1)$.
3. Explain why the calculation in (24.1) does not violate Stokes's Theorem.
4. If you are not already familiar, check the details of the correspondence (24.2).
5. Try to cook up an explicit smooth isomorphism between $\mathscr{O}(p)$ and $\mathscr{O}(q)$ for $p, q \in \Sigma$ connected.

25. $\bar{\partial}$-operator, Chern connection, integrability (4/21)

25.1. **$\bar{\partial}$-operators on bundles.** Recall that on a complex manifold, $\mathbb{C}$-valued differential forms split as:

$$\Omega^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Omega^{p,q},$$

where

$$\Omega^{p,q} = \left\langle dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q} \right\rangle.$$

If f is a $\mathbb{C}$-valued function, then

$$df = \partial f + \bar{\partial} f = \frac{\partial f}{\partial z^i} dz^i + \frac{\partial f}{\partial \bar{z}^j} d\bar{z}^j.$$

We extend ∂ and $\bar{\partial}$ to forms as follows. If

$$\omega = \frac{1}{p!q!} \omega_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} dz^{i_1} \wedge \cdots dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}$$

then

$$\bar{\partial}\omega := \frac{1}{p!q!} \frac{\partial \omega_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q}}{\partial \bar{z}^i} d\bar{z}^i \wedge dz^{i_1} \wedge \cdots dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge \bar{z}^{j_q}$$

and $\partial\omega$ is similar. As for functions,

$$d\omega = \partial\omega + \bar{\partial}\omega \in \Omega^{p+1,q} \oplus \Omega^{p,q+1}.$$

The Leibniz rule holds:

$$\bar{\partial}(\alpha \wedge \beta) = \bar{\partial}\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge \bar{\partial}\beta. \tag{25.1}$$

Moreover,

$$0 = d^2 = (\partial + \bar{\partial})^2 = \partial^2 + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\partial}^2.$$

Examining the types of the LHS, we see that

$$\partial^2 = 0 = \bar{\partial}^2. \tag{25.2}$$

and

$$\partial\bar{\partial} = -\bar{\partial}\partial. \tag{25.3}$$

Definition 25.1. Let E be a complex vector bundle on a complex manifold M. A **$\bar{\partial}$-operator** on E is a first-order differential operator

$$\bar{\partial} : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$$

satisfying the Leibniz rule (25.1), as well as

$$\bar{\partial}^2 = 0.$$

Recall that a holomorphic structure $\mathcal{E}$ on E is given by local holomorphic frames $\{f_\mu\}$ such that transition functions, defined by

$$f^a_\mu = g_{ab}{}^\nu{}_\mu f^b_\nu,$$

are all holomorphic.

Lemma 25.2. *For any holomorphic bundle $\mathcal{E}$, the $\bar{\partial}$-operator on M extends uniquely to a $\bar{\partial}$-operator on E, denoted $\bar{\partial}_{\mathcal{E}}$, such that*

$$\bar{\partial}_{\mathcal{E}} s = 0$$

for all holomorphic sections $s \in \Gamma(\mathcal{E})$.

Proof. Choose any local holomorphic frame $\{f_\mu\}$. Given $\omega = \omega^\mu \otimes f_\mu \in \Omega^{p,q}(E)$, define

$$\bar{\partial}_{\mathcal{E}}(\omega^\mu \otimes f_\mu) := \bar{\partial}\omega^\mu \otimes f_\mu.$$

For $s = s^\mu f_\mu$ a holomorphic section, we have

$$\bar{\partial}_{\mathcal{E}}(s^\mu f_\mu) = \bar{\partial} s^\mu \otimes f_\mu = 0$$

because all the s^μ's are holomorphic. So this satisfies the requirement, as long as it is well-defined. To show that $\bar{\partial}_{\mathcal{E}}$ is well-defined, suppose $f'_\nu = g^\mu{}_\nu f_\mu$ is another holomorphic frame, and compute:

$$\begin{aligned}
\bar{\partial}_{\mathcal{E}}(\omega^\nu \otimes f'_\nu) &= \bar{\partial}_{\mathcal{E}}(\omega^\nu \otimes g^\mu{}_\nu f_\mu) \\
&= \bar{\partial}_{\mathcal{E}}(g^\mu{}_\nu \omega^\nu \otimes f_\mu) \\
&= (\bar{\partial}(g^\mu{}_\nu) \wedge \omega^\nu + g^\mu{}_\nu \bar{\partial}\omega^\nu) \otimes f_\mu \\
&= \bar{\partial}\omega^\nu \otimes g^\mu{}_\nu f_\mu \\
&= \bar{\partial}\omega^\nu \otimes f'_\nu.
\end{aligned}$$

So our definition would have been the same in a different holomorphic frame. □

25.2. **Hermitian bundles and the Chern connection.** Recall that a Hermitian metric $\langle , \rangle$ on E is a smooth $\mathbb{C}$-valued bilinear form on the fibers, satisfying

$$\langle \lambda s, t \rangle = \bar{\lambda} \langle s, t \rangle$$
$$\langle s, \lambda t \rangle = \lambda \langle s, t \rangle$$
$$\langle t, s \rangle = \overline{\langle s, t \rangle}.$$

Definition 25.3. A **Hermitian vector bundle** is a holomorphic vector bundle with a Hermitian metric. A connection A on a Hermitian vector bundle is **compatible** with the Hermitian(/holomorphic) structure if

(1) A is unitary. Equivalently, A is compatible with the metric, meaning
$$X \langle s, t \rangle = \langle (D_A s)(X), t \rangle + \langle s, (D_A t)(X) \rangle$$
for any vector field X.

(2) $\bar{\partial}_A := \pi_{0,1} \circ D_A = \bar{\partial}_{\mathcal{E}}$.

Lemma 25.4. *Every Hermitian vector bundle has a unique compatible connection, called the* ***Chern connection****.*

Proof 1. Let $\{e_\alpha\}$ be a local unitary frame for E. Define $a \in \Omega^{0,1}(\mathrm{End}(E))$ by
$$\bar{\partial}_{\mathcal{E}} e_\alpha = a^\beta{}_\alpha e_\beta$$
and let
$$A = -a^* + a.$$
Recall from the third item in Example 15.2 that a connection is unitary if and only if its connection matrix *in a unitary frame* is skew-hermitian. Note that
$$A^* = -a + a^* = -A,$$
so our connection is indeed unitary. Moreover, by definition, we have
$$D_A^{0,1} e_\alpha = a^\beta{}_\alpha e_\beta = \bar{\partial}_{\mathcal{E}} e_\alpha.$$
It's also clearly unique, so well-defined globally. □

Proof 2. Let $\{f_\mu\}$ be a local holomorphic frame for $\mathcal{E}$.[25] The metric tensor is the section of $\bar{E}^* \otimes E^*$ with local components
$$H_{\bar{\mu}\nu} := \langle f_\mu, f_\nu \rangle .$$
Observe that
$$\overline{H_{\bar{\mu}\nu}} = H_{\bar{\nu}\mu}.$$
Let $H^{\nu\bar{\mu}}$ be the inverse matrix of $H_{\bar{\mu}\nu}$, satisfying
$$H^{\kappa\bar{\mu}} H_{\bar{\mu}\nu} = \delta^\kappa{}_\nu.$$
We want to find A such that $D_A^{0,1} f_\mu = 0$. This is true iff $A^{0,1} = 0$ in this (holomorphic) frame, so we must have $A = A^{1,0}$. Letting $D_A f_\mu = A^\nu{}_\mu f_\nu$, the compatibility condition reads:
$$dH_{\bar{\mu}\nu} = \langle A^\kappa{}_\mu f_\kappa, f_\nu \rangle + \langle f_\mu, A^\kappa{}_\nu f_\kappa \rangle = \overline{A^\kappa{}_\mu} H_{\bar{\kappa}\nu} + H_{\bar{\mu}\kappa} A^\kappa{}_\nu.$$

[25]Since we are making a different choice of frame, the connection form A in Proof 2 will be different from the connection form A in Proof 1.

Consequently, A must satisfy

$$\begin{aligned}\partial H_{\bar\mu\nu} &= H_{\bar\mu\kappa}A^\kappa{}_\nu\\ \bar\partial H_{\bar\mu\nu} &= \overline{A^\kappa{}_\mu}H_{\bar\kappa\nu}.\end{aligned}$$

Multiplying by the inverse matrix in the first equation, we have

$$H^{\kappa\bar\mu}\partial H_{\bar\mu\nu} = A^\kappa{}_\nu.$$

If we take this as our definition, then we can compute

$$\begin{aligned}\overline{A^\kappa{}_\mu}H_{\bar\kappa\nu} &= \overline{H^{\kappa\bar\lambda}\partial H_{\bar\lambda\mu}}H_{\bar\kappa\nu}\\ &= H^{\lambda\bar\kappa}\bar\partial H_{\bar\mu\lambda}H_{\bar\kappa\nu}\\ &= \bar\partial H_{\bar\mu\lambda}\delta^\lambda{}_\nu\\ &= \bar\partial H_{\bar\mu\nu}.\end{aligned}$$

So our choice also solves the second equation. □

Proposition 25.5 (Corollary of Proof 2 above). *Writing $H = (H_{\bar\mu\nu})$ for the metric and A for the connection form in a local holomorphic frame, we have the following local formula for the Chern connection*

$$A = H^{-1}\partial H = A^{1,0}$$

and its curvature

$$F_A = \bar\partial A.$$

In particular, the curvature of A is of type (1,1).

Proof. The formula for the connection was derived in Proof 2 of the previous Lemma. For the curvature, we have

$$\begin{aligned}F_A &= dA + A\wedge A = \partial A + \bar\partial A + A\wedge A\\ &= -H^{-1}\partial H H^{-1}\wedge\partial H + H^{-1}\partial^2 H + \bar\partial A + H^{-1}\partial H\wedge H^{-1}\partial H\\ &= \bar\partial A \in \Omega^{1,1}(\mathrm{End}E).\end{aligned}$$

□

Remark 25.6. We can obtain an even nicer local formula for the trace of the curvature:

$$\operatorname{Tr} F_A = \bar\partial \operatorname{Tr} H^{-1}\partial H = \bar\partial\partial\log\det H.$$

This formula is important in complex geometry.

25.3. **Integrability.** We showed above that any Hermitian vector bundle (*i.e.* a holomorphic bundle with metric) comes with a $\bar\partial$-operator and a unitary connection for which this $\bar\partial$-operator is just $\bar\partial_A = D_A^{0,1}$.

Conversely, we have the following answer to Question 3 in §2.2 above.

Theorem 25.7. *Any $\bar{\partial}$-operator on a complex bundle E, per Definition 25.1, gives rise to a holomorphic structure $\mathcal{E}$ on E.*

Equivalently, any unitary connection A on E with curvature $F_A \in \Omega^{1,1}(\mathrm{End}E)$ defines a holomorphic structure on E. (In particular, if $n = 1$, then $\Omega^{0,2} = 0 = \Omega^{2,0}$, so any unitary connection will do.)

Note: The statements are equivalent because $F_A = D_A^2$ is of type $(1,1)$ if and only if the $(0,2)$ part of F_A,

$$\left(D_A^{0,1}\right)^2 : \Omega^{0,0}(E) \to \Omega^{0,2}(E),$$

vanishes, *i.e.*, $D_A^{0,1} = \bar{\partial}_A$ defines a $\bar{\partial}$-operator. In this case, since A is unitary, the $(2,0)$ part must also vanish.

Proof. We'll prove the result for $n = 1$ and $M = \Sigma$ a Riemann surface. The higher-dimensional case requires only a bit more thought.

Choose a metric $\langle , \rangle$ on E and let A be the Chern connection for the given $\bar{\partial}$-operator on E, which is then just $\bar{\partial}_A$. Given $x_0 \in \Sigma$, we must construct a frame $\{f_\mu\}$ near x_0 such that $\bar{\partial}_A f_\mu = 0$. (Exercise: use the Leibniz rule to show that the transition functions $g^\mu{}_\nu$ between these local frames must be holomorphic.)

Start with any unitary frame $\{e_\alpha\}$ near x_0, and let

$$\bar{\partial}_A e_\alpha = a^\beta{}_\alpha d\bar{z} \otimes e_\beta,$$

where $a^\beta{}_\alpha \in C^\infty_{\mathbb{C}}$. We want to construct a new frame

$$f_\mu = g^\beta{}_\mu e_\beta,$$

with $g^\beta{}_\mu(z)$ invertible, such that

$$0 = \bar{\partial}_A f_\mu = \left(\frac{\partial g^\beta{}_\mu}{\partial \bar{z}} + a^\beta{}_\alpha g^\alpha{}_\mu \right) d\bar{z} \otimes e_\beta.$$

Equivalently, in matrix notation,

$$(*) \qquad \frac{\partial g}{\partial \bar{z}} + a \cdot g = 0.$$

We now make several reductions. First of all, because we're only looking for a local frame, it suffices to solve $(*)$ over a small disk $D_r \ni x_0$ for some $r > 0$. If we pull back along the map

$$\begin{aligned} D_1 &\to D_r \\ z &\mapsto r \cdot z \end{aligned}$$

then

$$\begin{aligned} \frac{\partial}{\partial \bar{z}} &\rightsquigarrow \frac{1}{r}\frac{\partial}{\partial \bar{z}} \\ a(z) &\rightsquigarrow a(r \cdot z) \end{aligned}$$

so $(*)$ becomes

$$\frac{1}{r}\frac{\partial g}{\partial \bar{z}} + a(rz)g = 0$$

or equivalently
$$\frac{\partial g}{\partial \bar z} + ra(rz)g = 0.$$
Moreover, since we are only interested in solving $(*)$ in a neighborhood, we can choose φ a smooth cutoff with $\varphi|_{D_{r/2}} \equiv 1$, $\operatorname{supp}\varphi \subset D_r$, and replace a by $\varphi \cdot a$. In this way, after rescaling, we are free to assume that
$$\operatorname{supp} a \subset D_1$$
and
$$\sup_{\mathbb{C}} |a| < \eta,$$
with $\eta > 0$ arbitrarily small.

We now recall the **Cauchy kernel**. Given $\theta(z)$ a compactly supported (matrix-valued) function on $\mathbb{C}$, the function
$$(L\theta)(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\theta(w)}{w - z} dw \wedge d\bar w$$
satisfies
$$\frac{\partial}{\partial \bar z}(L\theta(z)) = \theta(z).$$
(Exercise: prove this or see Lemma 1.6.3 of AW's notes on complex manifolds.) In addition, supposing that $\operatorname{supp}\theta \subset D_1$, we have the following estimate:
$$|L\theta(z)| \le \frac{\sup|\theta|}{2\pi}\int_{D_1} \frac{1}{|w|} dV_w = \frac{\sup|\theta|}{2\pi}\cdot 2\pi \int_0^1 \frac{1}{r} r dr = \sup|\theta|.$$
To solve $(*)$, we can take $g = \mathbf{1} + h$, so g solves $(*)$ if and only if h solves
$$\frac{\partial h}{\partial \bar z} = -a(1 + h).$$
Defining the integral operator
$$T(h) = L(-a(1 + h)),$$
this last equation is equivalent to the fixed-point equation
$$T(h) = h.$$
Since a is supported in the unit disk, the above estimate on L gives
$$\begin{aligned}\sup_{\mathbb{C}} |T(h_1) - T(h_2)| &= \sup_{\mathbb{C}} |L(-a(1 + h_1 - (1 + h_2)))| \\ &\le \sup|a(h_2 - h_1)| \\ &\le \eta \sup|h_2 - h_1|.\end{aligned}$$
Hence, for $\eta < 1$, T gives a contraction mapping
$$L^\infty(\mathbb{C}) \to L^\infty(\mathbb{C}).$$
By the Contraction Mapping Theorem, there exists a unique h such that $g = 1 + h$ solves $(*)$. Also, since $|T(h)| = |h| \le \eta(1 + |h|)$ (by the above calculation), assuming that $\eta < \frac{1}{2}$, $|h| < 1$ and g is invertible. Lastly, if follows by bootstrapping from the fixed-point equation that h is actually smooth.

The case $n > 1$ is proved by induction: see pp. 52-53 of Donaldson-Kronheimer. □

25.4. **Exercises.**

1. Use the Leibniz rule to show that $\bar{\partial}_A^2 = 0$ then for local frames $\{f_\mu\}, \{f'_\nu\}$ with $\bar{\partial}_A f_\mu = 0 = \bar{\partial}_A f'_\nu$, the transition functions $g^\mu{}_\nu$ must be holomorphic.
2. Prove the formula for the Cauchy Kernel (or see Lemma 1.6.3 of my complex manifolds notes).
3. Show that for any holomorphic section s of a holomorphic line bundle with metric, we have the local formula $\bar{\partial}\partial \log |s|_h^2 = \bar{\partial}\partial \log h$, where h is the local component of the metric in a holomorphic frame.
4. Use the formula of Remark 25.6 and the previous exercise to show that the degree of the divisor of zeroes of a (nontrivial) holomorphic section of a line bundle on a Riemann surface is equal to the degree. Conclude that a holomorphic bundle with negative degree can have no nontrivial holomorphic sections.

26. Action of the complexified gauge group (4/26)

26.1. **Isomorphism of holomorphic structures.** Recall that if $E \to \Sigma$ is a bundle over a Riemann surface, then a holomorphic structure $\mathcal{E}$ on E is equivalent to an operator $\bar{\partial}_{\mathcal{E}}$ such that $\bar{\partial}_{\mathcal{E}} s = 0$ for all $s \in \Gamma(\mathcal{E})$ holomorphic, the Leibniz rule holds, and $\bar{\partial}_{\mathcal{E}}^2 = 0$.

Fix a Hermitian metric $\langle , \rangle$ on E.

- Given a holomorphic structure $\bar{\partial}_{\mathcal{E}}$ on E, the **Chern connection** is the unique unitary connection A such that
$$\bar{\partial}_{\mathcal{E}} = \bar{\partial}_A : \Omega^0(E) \overset{D_A}{\to} \Omega^1(E) \overset{\pi_{0,1}}{\to} \Omega^{0,1}(E).$$
- Given a unitary connection A on E, $\bar{\partial}_A =: \bar{\partial}_{\mathcal{E}}$ defines a holomorphic structure on E (by Theorem 25.7); for this item we want to be on a Riemann surface to avoid requiring $\bar{\partial}_A^2 = 0$.

In this way, we have a complete equivalence between holomorphic structures and unitary connections on a fixed smooth bundle with metric over a Riemann surface. The goal will be to use this equivalence to describe the set of isomorphism classes of holomorphic structures on E.

Definition 26.1. Let $\mathcal{E}$ and $\mathcal{F}$ be two holomorphic structures on the same underlying smooth complex bundle E. We say that $\mathcal{E}$ and $\mathcal{F}$ are **isomorphic**, and write $\mathcal{E} \cong \mathcal{F}$, if there is an automorphism $g \in \mathrm{Aut}_{\mathbb{C}}(E)$ such that for each holomorphic section $s \in \Gamma(\mathcal{E})$, $g(s) \in \Gamma(\mathcal{F})$.

Equivalently,

$$\bar{\partial}_{\mathcal{F}} g(s) = g \bar{\partial}_{\mathcal{E}}(s) \tag{26.1}$$

for general sections s, or

$$\begin{aligned} \bar{\partial}_{\mathcal{F}} &= g \circ \bar{\partial}_{\mathcal{E}} \circ g^{-1} \\ &= \bar{\partial}_{\mathcal{E}} - \bar{\partial}_{\mathcal{E}} g \, g^{-1}. \end{aligned} \tag{26.2}$$

We introduce the notation
$$\mathscr{G}^{\mathbb{C}} = \mathrm{Aut}_{\mathbb{C}}(E) \subset E \otimes_{\mathbb{C}} E^*.$$
This discussion has shown that two holomorphic structures are isomorphic if and only if they belong to the same orbit under the $\mathscr{G}^{\mathbb{C}}$-action given by (26.2).

26.1.1. *Transforming Chern connections.* Fix a Hermitian metric on E. Suppose that the Chern connections of $\mathcal{E}$ and $\mathcal{F}$ are A and B, respectively. Then
$$\bar\partial_{\mathcal{F}} = \bar\partial_B = \bar\partial_A \underbrace{-\bar\partial_A g \cdot g^{-1}}_{=:a}.$$
From Proof 1 of Lemma 25.4 (existence of the Chern connection), we must have
$$\partial_B = \partial_A - a^* = \partial_A + (\bar\partial_A g \cdot g^{-1})^*.$$
Note that this does not agree with the usual transformation rule for connections; unless $g \in \mathscr{G}$ is unitary, in which case we have:
$$(\bar\partial_A g \cdot g^{-1})^* = (g^{-1})^* \partial_A (g^*) = g \partial_A g^{-1} = -g g^{-1} \partial_A g g^{-1} = -\partial_A g\, g^{-1},$$
and the action is just the ordinary action of the gauge group. Hence, the action of $\mathscr{G}^{\mathbb{C}}$ on Chern connections is an extension of the action of $\mathscr{G}$ by gauge transformations! This yields a powerful analogy:

- $\mathscr{G} \rightsquigarrow G$ finite dimensional compact Lie group
- $\mathscr{G}^{\mathbb{C}} \rightsquigarrow G^c$ the complexification of G
- $\mathscr{G}^c \curvearrowright \mathscr{A} \rightsquigarrow G^c \curvearrowright V$ with V a finite-dimensional complex representation.
- $\{$isom. classes of holomorphic structures$\} \rightsquigarrow \{G^c$-orbits$\}$.

26.2. **Invariant theory.** Given a complex Lie group acting linearly on a vector space, $G^c \curvearrowright V$, it is a classical problem to describe the space of orbits.

Example 26.2. Let $G = \mathrm{U}(1) \subset \mathbb{C}^* = G^c$. This acts on $V = \mathbb{C}^2$ by
$$\lambda \cdot \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \lambda z \\ \lambda^{-1} w \end{pmatrix}.$$
The orbits of this action on $V \setminus 0$ can be listed as follows:
$$\begin{aligned} \{(z,w) \mid zw = c \neq 0\} \\ \{(z,0)\} \\ \{(0,w)\}. \end{aligned}$$
The second two orbits are problematic because
$$\overline{\{(\lambda z, 0)\}} \cap \overline{\{(0, \lambda^{-1} w)\}} \neq \varnothing.$$
This means (analytically) that the quotient will not be Hausdorff and (algebraically) that you can't separate these two orbits with a holomorphic function.

Definition 26.3. A nonzero orbit $G^c \cdot x$ is **semistable** if $\overline{G^c \cdot x} \not\ni 0$.

In the above example, the first set of orbits are semistable (indeed stable, meaning the stabilizer of a point is discrete), and the second two are unstable. In Mumford's geometric invariant theory, the **GIT quotient** of V is given by

$$V//G^c \text{``:=''} V^{ss}/G^c,$$

where the "" are because we also need to identify (strictly) semistable orbits whose orbit closures intersect. A fundamental theorem in GIT states that this gives us an algebraic variety. In the above example, the GIT quotient is $\mathbb{C}^*$, just coming from the constant c in the first set of orbits; after projectivization, this is just a single point.

The task is then to understanding the properties (for instance, the topology) of $V//G^c$. There is a powerful approach to this problem via differential (in particular, symplectic) geometry, which is easiest to illustrate with the same example as above.

Example 26.4 (Example 26.2, continued)**.** Note that $V = \mathbb{C}^2$ has a symplectic form

$$\frac{i}{2}(dz \wedge d\bar{z} + dw \wedge d\bar{w}).$$

The action of $\mathrm{U}(1)$ on V preserves this form. There exists a moment map

$$\mu : V \to \mathrm{Lie}(\mathrm{U}(1)) = i\mathbb{R},$$

satisfying

$$\langle d\mu_x(-), \xi \rangle = \omega(L_x\xi, -),$$

where

$$L_x\xi := \frac{d}{dt}\Big|_{t=0} \exp(t\xi)x.$$

Here, up to constants, the moment map is given by

$$\mu\begin{pmatrix} z \\ w \end{pmatrix} = i(|z|^2 - |w|^2).$$

Notice (by inspection) that each stable orbit contains a zero of μ, unique up to the action of $\mathrm{U}(1)$. This points to the following general result:

Theorem 26.5 (Kempf-Ness)**.** *Assume that G^c acts on V as above, and the action of $G \subset G^c$ is unitary. Then, the G^c-orbit of $x \in V$ is semistable if and only if $\overline{G^c \cdot x} \cap \mu^{-1}(0) \neq \varnothing$.*

This theorem is basically trivial in the $\mathrm{U}(1)$ case, but it is quite nontrivial (and interesting) in the general case. It has the following corollary:

Corollary 26.6. *There is a homeomorphism*

$$V//G^c \cong \mu^{-1}(0)/G.$$

The quotient on the RHS is that of a much smaller set of points by a much smaller (indeed, a compact) group, so is potentially much easier to understand.

26.3. **Running with the analogy.** To complete the analogy, we need substitutes for the symplectic form, moment map, stability, *etc.*

26.3.1. *Symplectic form, complex structure, and metric.* Recall that

$$T_A\mathscr{A}_E \cong \Omega^1(\mathfrak{g}_E),$$

with A corresponding to a plus a reference connection. Define

$$\omega(a,b) := -\int_\Sigma \operatorname{Tr}(a \wedge b).$$

Since Tr is symmetric and $\wedge$ is alternating on 1-forms, we have

$$\omega(a,b) = \int_\Sigma \operatorname{Tr}(b \wedge a) = -\omega(b,a),$$

so this is alternating as desired. It's easy to see that ω is nondegenerate, and it's also closed for tautological reasons.

Next, we define the (almost)-complex structure

$$J : a \mapsto *a.$$

We have

$$J^2 = *^2 = (-1)^{1(2-1)} = -1.$$

We then get a metric for free:

$$\begin{aligned}(a,b) := \omega(a, Jb) &= -\int \operatorname{Tr} a \wedge *b \\ &= \int \langle a, b\rangle \, dV.\end{aligned}$$

26.3.2. *Moment map.*

Definition/Lemma 26.7. *A* ***moment map*** *for the action of $\mathscr{G}_E$ on $\mathscr{A}_E$ is given by*

$$\mu(A) := *F_A.$$

Proof. We need to check that

$$\langle d\mu(A)(a), \xi\rangle = \omega(L_A\xi, a)$$

for all $a \in T_A\mathscr{A}_E$ and $\xi \in \mathfrak{g}_E$.

We have

$$d\mu(A)(a) = \frac{d}{dt}\Big|_{t=0} * F_{A+ta} = *D_A a.$$

Also recall from Exercise 14.4.3. that the infinitesimal gauge action is given by

$$\frac{d}{dt}\Big|_{t=0}(\exp(-t\xi))(A) = D_A\xi = L_A\xi.$$

Hence,

$$\begin{aligned}\langle d\mu(A)(a), \xi\rangle = \int_\Sigma \langle *D_A a, \xi\rangle \, dV &= \int_\Sigma \operatorname{Tr}(\xi \cdot D_A a) \, dV \\ &= -\int_\Sigma \operatorname{Tr}(D_A\xi \wedge a) \, dV = \omega(L_A\xi, a).\end{aligned}$$

□

Note that zeroes of the moment map correspond to flat connections. Also note that the central elements of $\mathfrak{g}_E$ (for the adjoint action of $\mathscr{G}_E$) are precisely the constant multiples of $i\mathbf{1}_E$. You can always add a constant central element to the moment map. So in fact we should look for elements with $*F = i \cdot \mathit{const} \cdot \mathbf{1}_E$, *i.e.* projectively flat connections, since flat connections can't exist unless $c_1(E) = 0$. The constant is in turn determined by the the first Chern class, as we know from Theorem 19.2.

So, the analogy with invariant theory dictates that we should look for a projectively flat connection in each isomorphism class of holomorphic structures. This is exactly what we're set up to do!

Note that there's one last/crucial part of the analogy that we have not yet touched, which is the question of stability. This only enters for higher-rank bundles, as we'll discuss next time. As for the rank 1 case, we have the following refinement of Example 19.5 above.

Theorem 26.8 (Rank 1 case)**.** *Let L be a complex line bundle over Σ with a unitary connection A, and assume that* $\mathrm{Vol}(\Sigma) = 1$. *There exists a complexified gauge transformation $g \in \mathscr{G}^{\mathbb{C}}$ such that*

$$*F_{g(A)} = \frac{2\pi}{i} \deg(L).$$

In particular, any Hermitian (holomorphic) line bundle has a compatible *projectively flat connection.*

Note: Here, as above, $g(A)$ denotes the Chern connection of $g \circ \bar{\partial}_A \circ g^{-1}$.

Proof. We have

$$\begin{aligned} \bar{\partial}_{g(A)} &= \bar{\partial}_A - \bar{\partial}_A g\, g^{-1} \\ \partial_{g(A)} &= \bar{\partial}_A + (\bar{\partial}_A g g^{-1})^*. \end{aligned}$$

Note that g is a section of $\mathrm{End}_{\mathbb{C}}(L) \cong \underline{\mathbb{C}}$, since L is a line bundle, and A induces the product connection; so we may write $\partial_A = \partial$ and $\bar{\partial}_A = \bar{\partial}$. Taking $g = \exp(h)$, we obtain

$$\begin{aligned} \bar{\partial}_{g(A)} &= \bar{\partial}_A - \bar{\partial} h \\ \partial_{g(A)} &= \partial_A + (\bar{\partial} h)^* = \partial_A + \partial(h^*). \end{aligned}$$

So

$$F_{g(A)} = dA + d(-\bar{\partial} h + \partial(h^*)) = F_A - 2\partial\bar{\partial}\,\mathrm{Re}(h) \implies *F_{g(A)} = *F_A + 4i\Delta\,\mathrm{Re}(h).$$

Here we have used (25.2-25.3). Then

$$*F_{g(A)} + 2\pi i \deg(L) = *F_A + 2\pi i \deg(L) + 4i\Delta\,\mathrm{Re}(h) = 0$$

is solvable, by Chern-Weil, as in Example 19.5. $\square$

Remark 26.9. An analogous result holds for $(1,1)$ connections on line bundles over general compact Kähler manifolds, and follows from the $\partial\bar{\partial}$-lemma; see Griffiths and Harris, Proposition on p. 148.

26.4. **Exercises.**

1. Check the equivalent definition (26.1).
2. Show that the gauge transformation g constructed in Theorem 26.8 is unique up to an action of $\mathbb{R} \times \mathscr{G}$, where $\mathscr{G}$ is the unitary gauge group.

27. Holomorphic splitting, stability, Narasimhan-Seshadri Theorem (4/28-5/3)

27.1. **Plan of attack, main difficulty.** Let $\mathcal{E} \to \Sigma$ be holomorphic of rank $r \geq 1$. Our strategy for finding a compatible projectively flat connection will be:

(1) Fix A_0 with $\bar{\partial}_{A_0} = \bar{\partial}_{\mathcal{E}}$.
(2) Choose $g_i \in \mathscr{G}^{\mathbb{C}}$ such that $\{g_i(A_0)\}$ minimizes the Yang-Mills functional.
(3) Try to extract a limit B, which should be a Yang-Mills connection.
(4) See if $\mathcal{F}$, the holomorphic structure defined by $\bar{\partial}_B$, defines a holomorphic structure isomorphic to $\mathcal{E}$.

All of these will be relatively easy with the tools we have, except for the final step. Must $\mathcal{F}$ be isomorphic to $\mathcal{E}$? The kind of problem you might worry about is: $\mathcal{E}$ is not split, but $\mathcal{F}$ is. We can give the following explicit example.

Example 27.1 (From 5/3 class)**.** Let $\Sigma = \mathbb{CP}^1$. We work in the usual charts $(U_0, z), (U_1, w)$, with $z = w^{-1}$. We use the usual holomorphic frames for $\mathscr{O}(k)$, related by $\tau_1 = z^{-k}\tau_0$.

Let E be the underlying smooth bundle of $\mathscr{O}(-1) \oplus \mathscr{O}(1) \to \mathbb{CP}^1$. Define a $\bar{\partial}$-operator on E by

$$\bar{\partial}_A = \begin{pmatrix} \bar{\partial}_{\mathscr{O}(-1)} & a \\ 0 & \bar{\partial}_{\mathscr{O}(1)} \end{pmatrix},$$

where

$$a \in \Omega^{0,1}(\mathrm{Hom}(\mathscr{O}(1), \mathscr{O}(-1)) = \Omega^{0,1}(\mathscr{O}(-2))$$

is defined on (U_0, τ_0) by

$$a_0(z) = \frac{d\bar{z}}{(1 + |z|^2)^2}$$

and on (U_1, τ_1) by

$$a_1(w) = z^2 a_0(z) = \frac{-d\bar{w}}{\bar{w}^2(1 + |w|^{-1})^2} w^{-2} = \frac{-d\bar{w}}{(1 + |w|^2)^2}.$$

We claim that

$$s = \begin{cases} \left(\frac{-\bar{z}}{1+|z|^2}, 1\right), & \text{on } (U_0, \tau_0) \\ \left(\frac{-1}{1+|w|^2}, w\right), & \text{on } (U_1, \tau_1) \end{cases}$$

is a well-defined, non-vanishing, holomorphic section with respect to $\bar{\partial}_A$, *i.e.*, $\bar{\partial}_A s = 0$. This is easy to check (exercise).

Since s is non-vanishing, the holomorphic structure defined by $\bar{\partial}_A$ is *not* isomorphic to $\mathscr{O}(-1) \oplus \mathscr{O}(1)$. In fact, it's easy to write down a second, linearly independent section (exercise), which shows that the holomorphic structure defined by $\bar{\partial}_A$ is isomorphic to $\underline{\mathbb{C}}^2$. Hence, we have just written down the non-split extension[26]

$$0 \to \mathscr{O}(-1) \to \underline{\mathbb{C}}^2 \to \mathscr{O}(1) \to 0.$$

Now, let

$$g_t = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix},$$

so that

$$\begin{aligned} \bar{\partial}_{g_t(A_0)} = g_t \circ \bar{\partial}_A \circ g_t^{-1} &= g_t \begin{pmatrix} \bar{\partial}_{\mathscr{O}(-1)} & t^{-1} a \\ 0 & t^{-1} \bar{\partial}_{\mathscr{O}(1)} \end{pmatrix} \\ &= \begin{pmatrix} \bar{\partial}_{\mathscr{O}(-1)} & t^{-1} a \\ 0 & \bar{\partial}_{\mathscr{O}(1)} \end{pmatrix}. \end{aligned}$$

By definition, the isomorphism classes $[\bar{\partial}_{g_t(A)}] = [\bar{\partial}_A]$ are the same. However,

$$\lim_{t \to \infty} \bar{\partial}_{g_t(A)} = \begin{pmatrix} \bar{\partial}_{\mathscr{O}(-1)} & 0 \\ 0 & \bar{\partial}_{\mathscr{O}(1)} \end{pmatrix},$$

which is the split holomorphic structure on E. Hence, the limit of this 1-parameter family of $\bar{\partial}$-operators exists, but is not isomorphic to the rest of the family.

Note, however, that this exact pathology would not actually occur in our argument, because $\mathscr{O}(-1) \oplus \mathscr{O}(1)$ is not flat, while $\underline{\mathbb{C}}^2$ is, so we'd zoom right past the former in minimizing the Yang-Mills functional. This reflects the fact that $\underline{\mathbb{C}}^2$ is *semistable*, per the definition that we're about to give.

Remark 27.2. This setting is more subtle than finite-dimensional GIT, because there, stable orbits are by definition closed. Here, none of the orbits are closed, because every vector bundle over a Riemann surface has at least some proper subbundles.

27.2. **Stability.** The following condition will turn out to be the right one. Recall that the *slope* of $\mathcal{E}$ (or E) is given by

$$\mu(\mathcal{E}) = \mu(E) = \frac{\deg E}{\mathrm{rk}(E)}.$$

Definition/Lemma 27.3 (Mumford-Takemoto)**.** *A holomorphic vector bundle $\mathcal{E} \to \Sigma$ is said to be **stable** if for all proper holomorphic subbundles $\mathcal{S} \subset \mathcal{E}$,*

$$\mu(\mathcal{S}) < \mu(\mathcal{E})$$

or equivalently,

$$\mu(\mathcal{E}/\mathcal{S}) > \mu(\mathcal{E}).$$

*We say that $\mathcal{E}$ is **semistable** if the same statement holds with $\le$ replacing $<$.*

[26]There is a general theory of extensions of sheaves/bundles which you can read about in many places, for instance Donaldson-Kronheimer §10.2.1. The thing to keep in mind is that two extensions can be non-isomorphic, as extensions, even while the bundles in question are isomorphic. This is the case here with the family $g_t(A)$ for $t \neq 0$.

Proof of equivalence. Since all exact sequences split topologically, by Proposition 15.4, we have

$$\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)} = \frac{\deg(S) + \deg(E/S)}{\mathrm{rk}(S) + \mathrm{rk}(E/S)}.$$

For $b, d > 0$, it is trivial to see that

$$\frac{a}{b} < \frac{c}{d}$$

if and only if

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.$$

This yields the desired equivalence. □

Examples 27.4 (Some non-explicit examples; see Ch. 4 of *Algebraic surfaces and holomorphic vector bundles* by Friedman)**.**

1. $g = 0$: the Theorem of Grothendieck states that all holomorphic bundles on $\mathbb{CP}^1$ split holomorphically. Hence, by the lemma, the only stable bundles are line bundles (exercise). However, there are many semistable bundles, for instance $\underline{\mathbb{C}}^2$ (exercise).[27]
2. $g = 1$: Atiyah (1956) proved that all rank 2 bundles on an elliptic curve are extensions of line bundles. In particular, given $p \in \Sigma$, you can obtain a stable rank 2 bundle as the unique non-split extension of $\mathscr{O}(p)$ by $\mathscr{O}$. Since p is arbitrary, and you can always tensor a stable bundle with a line bundle, this gives lots of examples of rank-2 stable bundles on elliptic curves.
3. $g \geq 2$: given any line bundle $L \to \Sigma$ with $\deg L = -1$, there does exist a stable degree-0 extension
$$0 \to L \to V \to L^{-1} \to 0.$$
See Friedman for a proof.
4. It is a theorem of Lübke that $T\mathbb{CP}^n$ is stable. See the book of Okonek-Schneider-Spindler for much more on the subject. A theorem of Mehta and Ramanathan states that the restriction of a stable bundle to a generic curve in a sufficiently high power of an ample class is again stable. So, for a generic complete-intersection curve in $\mathbb{CP}^n$, the restriction of $T\mathbb{CP}^n$ is stable.
5. Flat unitary bundles have constant transition functions, so are automatically holomorphic. Recall that flat bundles of rank r are the same as representations of $\pi_1(\Sigma)$ in $\mathrm{U}(r)$, and it's not hard to show that these exist (exercise). The irreducible representations turn out to give stable bundles; this is the "if" direction of the big theorem that we're about to state.

Theorem 27.5 (Narasimhan-Seshadri, 1965; new proof by Donaldson, 1982)**.** *Let Σ be a compact Riemann surface with unit volume. An indecomposable Hermitian bundle $\mathcal{E} \to \Sigma$ (i.e. a holomorphic bundle with metric $\langle , \rangle$) is stable if and only if there exists a complexified gauge transformation $g \in \mathscr{G}_E^{\mathbb{C}}$ such that the isomorphic holomorphic structure on E defined by*

$$g \circ \bar{\partial}_{\mathcal{E}} \circ g^{-1}$$

[27]Notice that $\deg \mathscr{O}(-1) = -1$, so this was *not* a destabilizing subbundle of $\underline{\mathbb{C}}^2$ in the above example.

admits a compatible unitary connection A which is projectively flat, i.e., satisfies

$$*F_A = \frac{2\pi\mu(E)}{i}\mathbf{1}_E.$$

Moreover, g is unique up to the action of $\mathscr{G} \subset \mathscr{G}^{\mathbb{C}}$.

We'll prove this theorem in the next two sections.

Notice that for the case $c_1 = 0$, the theorem states that all stable bundles are (holomorphically) isomorphic to flat unitary bundles. Hence, the last example above is in fact the only example. This answers Question 4 in §2.2 above.

Also note that the theorem partly explains why algebraic geometers do not go to very great lengths to construct explicit examples of stable bundles on curves; we know exactly what they are. The task that has occupied many of them since 1965, however, is to describe the topology of this moduli space.

27.3. **Curvature of holomorphic subbundles and quotients.** We'll run through some material from Griffiths and Harris that's crucial to Donaldson's proof.

Suppose that

$$0 \to \mathcal{S} \to \mathcal{E} \to \mathcal{U} \to 0$$

is a SES of holomorphic vector bundles. Give E a metric $\langle , \rangle$ and compatible unitary connection A such that $\bar\partial_A = \bar\partial_{\mathcal{E}}$.

Smoothly, we can choose local unitary frames for E of the form

$$\{e_1, \ldots, e_s, e_{s+1}, \ldots, e_r\}$$

where $\{e_1, \ldots, e_s\}$ is a frame for S and $\{e_{s+1}, \ldots, e_r\}$ is a frame for $U \cong S^\perp$. (The last isomorphism is only smooth, not holomorphic.) If we think of A as a matrix of 1-forms, then

$$A = \begin{pmatrix} A_S & \beta \\ -\beta^* & A_U \end{pmatrix},$$

where

$$\beta = \pi_S D_A(e_{s+1}, \ldots, e_r)$$

is the **second fundamental form** of A. We have

$$\bar\partial_{\mathcal{E}} = \bar\partial_A|_S = \bar\partial_S$$

because $S \subset E$ is holomorphic, and

$$\bar\partial_U = \pi_U \circ \bar\partial_{\mathcal{E}}.$$

We have

$$0 = \pi_{S^\perp}\bar\partial_A(e_1, \ldots, e_s) = (-\beta^*)^{0,1},$$

so $\beta = \beta^{0,1}$ is a matrix of $(0,1)$-forms. Globally,

$$\beta \in \Omega^{0,1}(S \otimes U^*) \quad \text{ and } \quad \beta^* \in \Omega^{1,0}(U \otimes S^*).$$

If we use this to compute curvature, we obtain

$$\begin{aligned} F_A &= dA + A \wedge A \\ &= \begin{pmatrix} dA_S & d\beta \\ -d\beta^* & dA_U \end{pmatrix} + \begin{pmatrix} A_S \wedge A_S - \beta \wedge \beta^* & A_S \wedge \beta + \beta \wedge A_U \\ -\beta^* \wedge A_S - A_U \wedge \beta^* & -\beta^* \wedge \beta + A_U \wedge A_U \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} F_{A_S} - \beta \wedge \beta^* & D_{S\otimes U^*}\beta \\ -D_{U\otimes S^*}\beta^* & F_{A_U} - \beta^* \wedge \beta \end{pmatrix}.$$

On a Riemann surface,

$$\beta = \beta_{\bar{z}} d\bar{z} \quad \text{and} \quad \beta^* = (\beta_{\bar{z}})^* dz,$$

and

$$i(-\beta \wedge \beta^*) = -i\beta_{\bar{z}}\beta_{\bar{z}}^* \, d\bar{z} \wedge dz = 2\beta_{\bar{z}}\beta_{\bar{z}}^* d\mathrm{Vol}.$$

Consequently,

$$iF_A\Big|_S = F_{A_S} - i\beta \wedge \beta^* \geq iF_{A_S}$$

in the sense that the LHS is more positive (as a Hermitian matrix) relative to dVol than the RHS. This goes under the slogan:

curvature decreases in holomorphic subbundles.

Similarly, we have

$$iF_A\Big|_U = F_{A_U} - i\beta^* \wedge \beta \leq iF_{A_U},$$

which is to say:

curvature increases in holomorphic quotients.

Remark 27.6. There is a general notion of positivity for Hermitian vector bundles on complex manifolds, called "Griffiths positivity." The line bundle case is well-known if you've studied the Kodaira embedding theorem: over a Riemann surface, it's just saying that if you go around a counterclockwise loop at a point, then the holonomy on the fiber also goes counterclockwise.

27.4. Exercises.

1. Verify the claims in Example 27.1.
2. Show that a stable bundle is indecomposable, *i.e.*, cannot be holomorphically isomorphic to a direct sum of proper subbundles.
3. Show that $\underline{\mathbb{C}}^2 \to \mathbb{CP}^1$ is semistable.
4. Write down an irreducible representation of $\pi_1(\Sigma)$, for Σ compact of genus $g \geq 2$, into $\mathrm{SU}(2)$. Also prove that for $g = 1$, any representation is reducible.

28. Donaldson's proof, I (4/28-5/3)

We're now in a position to give Donaldson's proof (1982) of Theorem 27.5.

For the sake of simplicity, we will restrict our attention to the case $\mathrm{rk}(E) = 2$ and $\deg(E) = 0$. Stability in this case means that for all holomorphic sub-line-bundles $\mathcal{S} \subset \mathcal{E}$, $\deg(\mathcal{S}) < 0$, or equivalently, $\deg(\mathcal{E}/\mathcal{S}) > 0$. The equation to solve becomes $*F_A = 0$, or $F_A = 0$, so we are just looking for a flat connection compatible with the holomorphic structure.

Let A_0 be the Chern connection of $(\mathcal{E}, \langle\cdot,\cdot\rangle)$. Choose $g_i \in \mathscr{G}^{\mathbb{C}}$ such that

$$A_i := g_i(A_0)$$

is a minimizing sequence of the Yang-Mills energy over $\{g \in \mathscr{G}^{\mathbb{C}}\}$. Explicitly, this means

$$\mathcal{YM}(A_i) \to \inf_{g \in \mathscr{G}^{\mathbb{C}}} \mathcal{YM}(g(A_0)).$$

By Uhlenbeck's theorem, there exist $\sigma_i \in \mathscr{G}$ such that, after passing to a subsequence, we obtain weak L^2_1 convergence

$$\sigma_i(A_i) \rightharpoonup B, \tag{28.1}$$

and

$$\mathcal{YM}(B) \le \inf_{g \in \mathscr{G}^{\mathbb{C}}} \mathcal{YM}(g(A_0)).$$

(Exercise: justify the use of Uhlenbeck's theorem here.) We replace g_i by $\sigma_i g_i$, so that (28.1) becomes

$$A_i \rightharpoonup B.$$

Two questions now remain:

(1) Does $\bar\partial_B$ define a holomorphic structure isomorphic to that of $\bar\partial_{A_0}$?
(2) Is B projectively flat?

Let $\mathcal{E}$ and $\mathcal{F}$ be the holomorphic structures of $\bar\partial_A$ and $\bar\partial_B$, respectively. We need to compare them. This will be based on the following extremely cute observation.

Claim. A complexified gauge transformation $g \in \mathscr{G}^{\mathbb{C}} \subset \Omega^{0,0}(E \otimes E^*)$ is holomorphic, as a map

$$\mathcal{E} \to \mathcal{F},$$

if and only if

$$\bar\partial_{B \otimes A_0} g = 0.$$

Proof of claim. Let s be a holomorphic section of $\mathcal{E}$, equivalently,

$$\bar\partial_A s = 0 \overset{\text{loc}}{=} \bar\partial s + A \cdot s.$$

Locally, we have

$$\begin{aligned}\bar\partial_B(g(s)) &= \bar\partial g \cdot s + g \cdot \bar\partial s + B \cdot g \cdot s - g \cdot A \cdot s + g \cdot A \cdot s \\ &= (\bar\partial g + B \cdot g - g \cdot A) \cdot s + g \cdot \underbrace{(\bar\partial s + A \cdot s)}_{0} \\ &= \bar\partial_{B \otimes A}(g) \cdot s.\end{aligned}$$

Hence, the LHS vanishes for all s if and only if $\bar\partial_{B\otimes A}(g) = 0$. ◇

Lemma 28.1 (Key analytic lemma)**.** *There exists a nonzero holomorphic map*

$$\mathcal{E} \overset{\alpha}{\to} \mathcal{F}.$$

Proof. Assume the contrary. Then

$$\ker(\bar\partial_{B \otimes A_0}) = 0.$$

The operator $\bar\partial_{B\otimes A_0}$ is elliptic, so we have the strong elliptic estimate

$$\|g\|_{L^2_1} \le C \left\| \bar\partial_{B \otimes A_0} g \right\|.$$

(Compare with (21.2), for the operator $d \oplus d^*$.) Since we're in dimension less than four, we also have the Sobolev inequality

$$\|g\|_{L^4} \le C\,\|g\|_{L^2_1}.$$

We also know:

- $A_i \rightharpoonup B$ weakly in L^2_1, so $A_i \to B$ strongly in L^4
- $\bar\partial_{B\otimes A_0} - \bar\partial_{A_i\otimes A_0} = (B - A_i)_{0,1}$
- $\left\|\bar\partial_{B\otimes A_0} g - \bar\partial_{A_i\otimes A_0} g\right\| \le \|B - A\|_{L^4}\,\|g\|_{L^4}$ by Hölder.

Assembling the pieces, we have

$$\begin{aligned}\|g\|_{L^4} \le C\left\|\bar\partial_{B\otimes A_0} g\right\| &\le C\left\|\bar\partial_{A_i\otimes A_0} g + (\bar\partial_{B\otimes A_0} - \bar\partial_{A_i\otimes A_0})g\right\| \\ &\le C\left(\left\|\bar\partial_{A_i\otimes A_0} g\right\| + \|B - A_i\|_{L^4}\,\|g\|_{L^4}\right).\end{aligned}$$

Rearranging, we have

$$(1 - C\,\|B - A_i\|_{L^4})\,\|g\|_{L^4} \le C\left\|\bar\partial_{A_i\otimes A_0} g\right\|.$$

Choosing i sufficiently large, we may assume

$$\|B - A_i\|_{L^4} < \frac{1}{2C}.$$

This gives us

$$\|g\|_{L^4} \le C\left\|\bar\partial_{A_i\otimes A_0} g\right\|.$$

We now apply this estimate with $g = g_i$, to obtain

$$\|g_i\|_{L^4} \le C\left\|\bar\partial_{A_i\otimes A_0} g_i\right\| = 0,$$

since g_i is holomorphic between $\bar\partial_{A_0}$ and $\bar\partial_{A_i} = \bar\partial_{g_i(A_0)}$, so $g_i = 0$.

But $g_i \in \mathscr{G}^{\mathbb{C}}$, which contains only invertible sections, so this is absurd. □

It remains to show that if $\mathcal{E}$ is stable, then α is an isomorphism. There are two cases:

(1) Suppose α has full rank somewhere on $\mathcal{E}$. Then

$$\wedge^2\alpha : \wedge^2\mathcal{E} \to \wedge^2\mathcal{F}$$

is not identically zero. But

$$\deg \wedge^2\mathcal{E} = \deg\mathcal{E} = \deg E = 0 = \deg\mathcal{F} = \deg\wedge^2\mathcal{F},$$

where the first and last equalities are by definition. So

$$\deg(\mathrm{Hom}(\wedge^2\mathcal{E}, \wedge^2\mathcal{F})) = \deg(\mathcal{F}) - \deg(\mathcal{E}) = 0.$$

By Exercise 25.44., the holomorphic section $\wedge^2\alpha$ vanishes nowhere, so α has full rank everywhere. This completes the proof in the first case, leaving only:

(2) Suppose that α has full rank nowhere. Let $\mathcal{P} = \ker(\alpha)$ (*i.e.* the sheaf kernel), which a moment's thought shows to be a sub-line-bundle (exercise). We therefore have a short exact sequence

$$0 \to \mathcal{P} \to \mathcal{E} \to \mathcal{Q} \to 0, \tag{28.2}$$

where $\mathcal{Q} := \mathcal{E}/\mathcal{P}$. Since $\deg\mathcal{E} = 0$, we have

$$\deg\mathcal{Q} = d > 0$$

by stability, and

$$\deg \mathcal{P} = -d < 0,$$

since $\deg \mathcal{E} = 0$ by assumption.

Claim. There is a sub-line-bundle $\mathcal{M} \subset \mathcal{F}$ such that $\deg \mathcal{M} \geq d$.

Proof of claim. We have

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{P} \arrow[r] & \mathcal{E} \arrow[r] \arrow[d, "\alpha"] & \mathcal{Q} \arrow[r] \arrow[d, "\cong"] & 0 \\ & & \mathcal{F} & \mathcal{M}_0 := \operatorname{Im}(\alpha). \arrow[l] & \end{tikzcd}$$

As it stands, $\mathcal{M}_0$ is a subsheaf of $\mathcal{F}$, which is a subbundle where $\operatorname{rk}(\alpha) = 1$. Let $\{z_1, \ldots, z_n\} \subset \Sigma$ be the set of points where $\operatorname{rk}(\alpha) = 0$. Near z_i, we have

$$\alpha(z) = \begin{pmatrix} \alpha_{11}(z) & \alpha_{12}(z) \\ \alpha_{21}(z) & \alpha_{22}(z) \end{pmatrix} = (z - z_i)^{k_i} \begin{pmatrix} \tilde{\alpha}_{11}(z) & \tilde{\alpha}_{12}(z) \\ \tilde{\alpha}_{21}(z) & \tilde{\alpha}_{22}(z) \end{pmatrix}.$$

where $\tilde{\alpha}_{ij}(z_i) \neq 0$ for some i, j. Hence, $\tilde{\alpha}$ defines a map

$$\mathcal{M} := \mathcal{M}_0 \otimes \bigotimes_i \mathscr{O}(k_i z_i) \xrightarrow{\tilde{\alpha}} \mathcal{F}.$$

By construction, $\operatorname{rank}(\tilde{\alpha}) = 1$ everywhere, so $\mathcal{M} \subset \mathcal{F}$ is a subbundle. Also by construction, we have

$$\deg \mathcal{M} = \deg \mathcal{M}_0 + \sum_i k_i \geq d.$$

$\diamond$

We now have, in addition to (28.2), the short exact sequence of holomorphic vector bundles

$$0 \to \mathcal{M} \to \mathcal{F} \to \mathcal{N} \to 0, \tag{28.3}$$

where $\deg \mathcal{M} \geq d$ and $\deg \mathcal{N} \leq -d$. These two sequences will be used next time to prove two lemmas, both based on §27.3, which will complete the proof.

28.1. **Exercises.**

1. Justify the use of Uhlenbeck's compactness theorem to extract a weak L^2_1 limit from a minimizing sequence of $\mathcal{YM}$ over a Riemann surface.
2. Show that $\mathcal{P} = \ker \alpha$ is a sub-line-bundle of $\mathcal{E}$ in the situation above.

29. Donaldson's proof, II (5/5)

Lemma 29.1. *In the case that α has full rank nowhere, we have*

$$\mathcal{YM}(B) \geq 4\pi^2 d^2.$$

Proof. Recall our second exact sequence:

$$0 \to \mathcal{M} \to \mathcal{F} \to \mathcal{N} \to 0.$$

In a unitary frame,

$$B = \begin{pmatrix} B_{\mathcal{M}} & \beta \\ -\beta^* & B_{\mathcal{N}} \end{pmatrix}$$

so

$$F_B = \begin{pmatrix} F_{B_{\mathcal{M}}} - \beta \wedge \beta^* & D\beta \\ -D\beta^* & F_{B_{\mathcal{N}}} - \beta^* \wedge \beta. \end{pmatrix}$$

Let $f(z) := i * F_{\beta_{\mathcal{M}}}$. Then

$$\begin{aligned} \int |F_{B_{\mathcal{M}}} - \beta \wedge \beta^*|^2 &= \int |i * F_{B_{\mathcal{M}}} - i * (\beta \wedge \beta^*)|^2 d\mathrm{Vol} \\ &= \int |f(z) + |\beta|^2|^2 d\mathrm{Vol} \\ &\geq \left(\int (f(z) + |\beta|^2) d\mathrm{Vol} \right)^2 \\ &= \left(2\pi d + \int |\beta|^2 d\mathrm{Vol} \right)^2 \\ &\geq 4\pi^2 d^2. \end{aligned}$$

Similarly,

$$\int |F_{B_{\mathcal{N}}} - \beta^* \wedge \beta|^2 d\mathrm{Vol} \geq 4\pi^2 d^2.$$

Hence,

$$\mathcal{YM}(B) = \frac{1}{2} \int |F_B|^2 d\mathrm{Vol} \geq \frac{1}{2}(4\pi^2 d^2 + 4\pi^2 d^2) = 4\pi^2 d^2.$$

□

Lemma 29.2. *In the case that α has full rank nowhere, there exists $g \in \mathscr{G}^{\mathbb{C}}$ such that*

$$\mathcal{YM}(g(A_0)) < 4\pi^2 d^2.$$

To prove Lemma 29.2, we need a tiny bit of Hodge theory.

Proposition 29.3 (Kähler identities on a Riemann surface)**.**

(1) On $\Omega^{0,1}$, $\bar{\partial}_A^* = -i * \partial_A$.
(2) On $\Omega^{1,0}$, $\partial_A^* = i * \bar{\partial}_A$.

Proof. Let $\alpha \in \Omega^{0,0}$ and

$$\beta = \beta_{\bar{z}} d\bar{z} \in \Omega^{0,1}.$$

These forms pair as follows:

$$\langle \alpha, \beta \rangle = \bar{\alpha} \beta_{\bar{z}} d\bar{z}.$$

Hence,

$$d\langle\alpha,\beta\rangle = (D_z\bar{\alpha})\beta_{\bar{z}}dz\wedge d\bar{z} + \bar{\alpha}D_z\beta_{\bar{z}}dz\wedge d\bar{z}$$
$$= \left(\overline{D_{\bar{z}}\alpha}\beta_{\bar{z}} + \bar{\alpha}D_z\beta\bar{z}\right)dz\wedge d\bar{z}$$

We can reexpress the terms above:

$$\left\langle\bar{\partial}_A\alpha,\beta\right\rangle = \langle D_{\bar{z}}\alpha d\bar{z},\beta_{\bar{z}}d\bar{z}\rangle = 2\overline{D_{\bar{z}}\alpha}\beta_{\bar{z}}$$
$$\frac{i}{2}\partial_A\beta = \frac{i}{2}D_z\beta_{\bar{z}}dz\wedge d\bar{z} = D_z\beta_{\bar{z}}d\mathrm{Vol} \implies *\frac{i}{2}\partial_A\beta = D_z\beta_{\bar{z}}.$$

Consequently,

$$d\langle\alpha,\beta\rangle = -\frac{1}{2}\left(\left\langle\bar{\partial}_A\alpha,\beta\right\rangle + \langle\alpha, i*\partial_A\beta\rangle\right)dz\wedge d\bar{z}.$$

Integrating yields the claimed adjoint relation (1). We leave (2) as an exercise. □

Proof of Lemma 29.2. Recall our first exact sequence

$$0\to\mathcal{P}\to\mathcal{E}\to\mathcal{Q}\to 0.$$

Regard $E = P\oplus Q$ as smooth bundles. We first act by a block-diagonal complex gauge transformation

$$g = \begin{pmatrix} g_P & 0 \\ 0 & g_Q \end{pmatrix}$$

such that $g(A_0)_P$ and $g(A_0)_Q$ are both projectively flat. This is possible by Theorem 26.8. Set $A = g(A_0)$, which takes the form

$$A = \begin{pmatrix} A_P & \beta \\ -\beta^* & A_Q \end{pmatrix}$$

where

$$F_{A_P} = 2\pi di\omega$$
$$F_{A_Q} = -2\pi di\omega.$$

Here, ω is the Kähler from (volume form). Moreover, note that $\beta\not\equiv 0$ because $\mathcal{E}$ does not split holomorphically, by assumption.

The curvature is now given by

$$F_A = \begin{pmatrix} 2\pi id\omega - \beta\wedge\beta^* & D_A\beta \\ -D_A\beta^* & -2\pi id\omega - \beta^*\wedge\beta \end{pmatrix}.$$

Next, we act by

$$g = \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}.$$

This has the effect

$$\bar{\partial}_A \rightsquigarrow \bar{\partial}_A - \begin{pmatrix} 0 & \bar{\partial}_A\alpha \\ 0 & 0 \end{pmatrix}$$
$$\beta\rightsquigarrow\beta - \bar{\partial}_A\alpha.$$

By Hodge theory for $\bar\partial_A$, $\bar\partial_A\beta = 0$ implies that there exists an α such that

$$\bar\partial_A^*(\beta + \bar\partial_A\alpha) = 0.$$

We make make a further replacemeent

$$\beta \rightsquigarrow \beta + \bar\partial_A\alpha.$$

But by the last Proposition (Kähler identities), we have

$$\bar\partial_A\beta = 0 = \bar\partial^*\beta = -i * \partial_A\beta,$$

so

$$D_A\beta = \partial_A\beta + \bar\partial_A\beta = 0.$$

Hence, the off-diagonal blocks of the curvature now also vanish.

Finally, let

$$g_t = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$$

and put $A_t = g_t(A_t)$. As in Example 27.1, this has the effect of replacing β with $t\beta$. We therefore have

$$F_{A_t} = \begin{pmatrix} 2\pi i d\omega - t^2\beta\wedge\beta^* & 0 \\ 0 & -2\pi i\omega - t^2\beta^*\wedge\beta \end{pmatrix}.$$

We obtain

$$iF_{A_t} = \begin{pmatrix} -2\pi d + t^2|\beta|^2 & 0 \\ 0 & 2\pi d - t^2|\beta|^2 \end{pmatrix}\omega$$

and

$$\mathcal{YM}(F_{A_t}) = \int |2\pi d - t^2|\beta|^2|^2 d\mathrm{Vol} < 4\pi^2 d^2$$

for small t. □

Corollary 29.4. *The map α is an isomorphism.*

Proof. Otherwise, the two lemmas contradict one-another. □

A few points remain:

- $F_B = 0$
- Uniqueness
- Flat $\implies$ (semi)stable.

Since it's the last day of class, we'd better content ourselves with the first item.

Since $\mathcal{E} \cong \mathcal{F}$ is stable, the only endomorphisms of $\mathcal{E}$ are scalar multiples of the identity (exercise). Equivalently,

$$\langle \mathbf{1} \rangle = \ker D_B|_{\mathrm{End}E} = \ker D_B^* D_B|_{\mathrm{End}E}.$$

Since

$$\int \operatorname{Tr} F_B = \deg E = 0 = \int \langle \mathbf{1}, *F_B \rangle \, d\mathrm{Vol},$$

there is a nonzero Hermitian matrix h satisfying

$$D_B^* D_B h = -i * F_B.$$

Let $g_t = \exp(th)$ and $B_t = g_t(B)$. Computing as in the proof of the rank-1 case (Theorem 26.8), the first variation is

$$\delta_t B_t = \partial_B h - \bar\partial_B h.$$

Observe that

$$*dz = id\bar z$$
$$*d\bar z = -idz,$$

so

$$*D_B h = *(\partial_B h + \bar\partial_B h) = i(\bar\partial_B - \partial_B)h.$$

Multiplying by i, we obtain

$$\delta_t B_t = i * D_B h.$$

Recall our formula for the first variation of the Yang-Mills functional:

$$\delta_t \mathcal{YM}(B_t) = \int \langle \delta_t B_t, D_B^* F_B \rangle \, d\mathrm{Vol}.$$

Since $D_B^* = - * D_B *$ and $*$ is an isometry, this gives

$$\begin{aligned}
\delta_t \mathcal{YM}(B_t) &= -\int \langle i D_B h, D_B * F \rangle \, d\mathrm{Vol} \\
&= -\int \langle i D_B^* D_B h, *F_B \rangle \, d\mathrm{Vol} \\
&= -\int \langle *F_B, *F_B \rangle \, d\mathrm{Vol} \\
&= -\|F_B\|^2 .
\end{aligned}$$

But B is a minimizer, so

$$0 \le \delta_t \mathcal{YM}(B_t) = -\|F_B\|^2.$$

Therefore B is flat!

Remark 29.5. In general dimension, one replaces $*F$ with $\Lambda_\omega F$, where Λ_ω is the adjoint of the Lefschetz operator. This gives the Hermitian Yang-Mills equation:

$$\Lambda_\omega F + 2\pi i \mu(E) \mathbf{1}_E = 0.$$

Theorem 29.6 (Donaldson-Uhlenbeck-Yau)**.** *An indecomposable holomorphic bundle on a compact Kähler manifold is stable if and only if it admits a Hermitian Yang-Mills connection.*

In complex dimension two/real dimension four, and for a bundle with $\deg E = 0$, it turns out that you're Hermitian Yang-Mills ($\Lambda_\omega F = 0 = F^{0,2} = \overline{F^{2,0}}$) if and only if you're an instanton ($F_A^+ = 0$). For this reason, the DUY theorem plays a fundamental role in Donaldson theory. In higher dimensions, it has yet to achieve its full potential.

29.1. **Exercises.**

1. Prove Proposition 29.3(2).
2. Prove that a stable bundle is simple, *i.e.*, has no automorphisms other than constant multiples of the identity. (Hint: try the usual Schur's lemma argument.)